Examples of Mass Energy Equivalence

Example: A cube of density ρ_o in rest frame moving with velocity v with respect to observer parallel to one of its edge. What is density measured by observer?

Solution: In rest frame $\frac{m_0}{V_0}=\rho_0$ and from moving frame $\rho=\frac{\text{mass}}{\text{volume}}$, where m is relativistic mass given by, $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ and v is relativistic volume, $V=l_0 l_0\cdot l_0\sqrt{1-\frac{v^2}{c^2}}$

$$\rho=\frac{\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}{V_0\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{V_0\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}\Rightarrow\rho=\frac{\rho_0}{1-\frac{v^2}{c^2}}\qquad\left[\because\frac{m_0}{V_0}=\rho_0\right]$$

Example: Show that the rest mass of particle of momentum p and kinetic energy T is given by $m_0=\frac{p^2c^2-T^2}{2Tc^2}$

Solution:

$$E=E_K+E_0\Rightarrow E=T+m_0c^2\qquad\text{(i)}$$

$$E^2=p^2c^2+m_0^2c^4\qquad\text{(ii)}$$

Thus, from (i) and (ii), $m_0=\frac{p^2c^2-T^2}{2Tc^2}$

Example: In the laboratory frame a particle P at rest mass m_0 is moving in the positive x - direction with speed $\frac{5c}{19}$. It approaches an identical particle Q which is moving in the negative x - direction with a speed of $\frac{2c}{5}$.

(a) What is speed of the particle P in the rest frame of the particle Q.

(b) What is Energy of the particle P in the rest frame of the particle Q.

Solution: (a) assume Q is S frame so laboratory can be treated as S' frame so

$v_{L,Q} = v_{S',S} = v = -\left(-\frac{2}{5}c\right)\hat{i} \Rightarrow v = \frac{2}{5}c\hat{i}$,

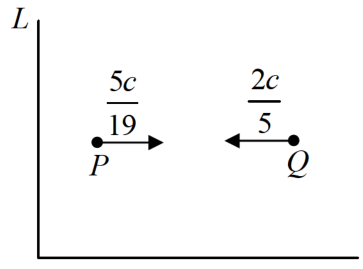


Figure 2.4

Speed of P with respect to lab is $v_{P,L} = v_{P,S'} = u'_x = \frac{5}{19}c$

$$u'_x = \frac{5}{19}c,\; v = \frac{2}{5}c,\; u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{3}{5}c$$

(b) $v_{P,Q} = u_x = \frac{3c}{5}$, $E = \frac{m_0c^2}{\sqrt{1 - \frac{v_1^2}{c^2}}} = \frac{5}{4}m_0c^2$

Example: The relativistic mass m of moving particle is $\frac{2m_0}{\sqrt{3}}$, where m_o is its rest mass. Then what is linear momentum of particle?

Solution: and mass, $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \frac{2m_0}{\sqrt{3}} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow v = \frac{c}{2}$

Momentum, $p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$ $p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{1}{4}}} \times \frac{1}{2}c \Rightarrow p = \frac{m_0 c}{\sqrt{3}}$

Example: A particle of rest mass m_0 moving with momentum $\frac{3m_0c}{4}$.

(a) Find the total relativistic energy of particle.

(b) Find the speed of particle and verify momentum $p = \frac{3}{4}m_0c$

(c) Find the kinetic energy of particle.

Solution: (a) Momentum is $p=\dfrac{3m_0c}{4}$ and rest mass energy is m_0c^2

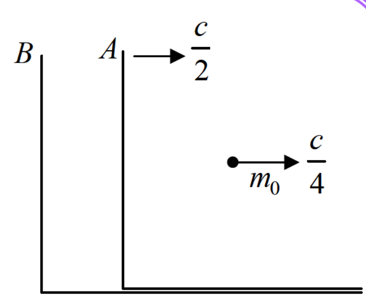
$$E=\sqrt{p^2c^2+m_0^2c^4}=E=\sqrt{\frac{9}{16}m_0^2c^4+m_0^2c^4}=\frac{5}{4}m_0c^2$$

(b) $E=mc^2\Rightarrow E=\dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}=\dfrac{5}{4}m_0c^2=\dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}\Rightarrow 1-\dfrac{v^2}{c^2}=\dfrac{16}{25}\Rightarrow v=\dfrac{3}{5}c$

$$p=mv=\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0\cdot\frac{3}{5}c}{\sqrt{1-\frac{9}{25}}}=\frac{3}{4}m_0c$$

(c) Relativistic kinetic energy $T=E-m_0c^2=\dfrac{5}{4}m_0c^2-m_0c^2=\dfrac{m_0c^2}{4}$

Example: A particle of rest mass m_0 moving with speed $\dfrac{c}{4}$ with respect to frame A in positive $\hat{x}$ direction .where frame B is moving with velocity $\dfrac{c}{2}$ with respect frame A in positive $\hat{x}$ direction .



(a) Find the relativistic energy of particle measured by observer attached with frame A

(b) Find the relativistic energy of particle measured by observer attached to frame B

Solution: (a) The rest mass of particle is m_0. If observer is attached to frame A the speed of mass with respect to mass is $\dfrac{c}{2}$, so relativistic mass $m=\dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}=\dfrac{m_0}{\sqrt{1-\dfrac{1}{16}}}=\sqrt{\dfrac{16}{15}}m_0$

so relativistic energy is $E=mc^2=\sqrt{\dfrac{16}{15}}m_0c^2$

(b) It is given $v_{m,A}=\dfrac{c}{4}\hat{x}$, $v_{A,B}=\dfrac{c}{2}\hat{x}$ $v_{m,B}=?$

We assume frame B is equivalent to S frame so then frame A is equivalent to S' frame.

$$v_{A,B}=v_{S',S}=v=\frac{c}{2}\hat{x},\ v_{m,A}=v_{m,S'}=u_x'=\frac{c}{4}\hat{x}\ ,\ v_{m,B}=v_{m,S}=u_x=?$$

Using formula of relative speed $u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{\frac{c}{2} + \frac{c}{4}}{1 + \frac{c}{4} \cdot \frac{c}{2} \cdot \frac{1}{c^2}} = \frac{2c}{3}$

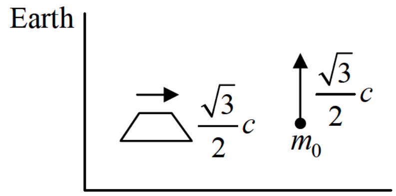
So, observer attached to frame B will measured that mass is moving the speed $\frac{2c}{3}$.

The relativistic mass measured by observer B is $m = \frac{m_0}{\sqrt{1 - \frac{|v_{m,A}|^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{4}{9}}} = \frac{3m_0}{\sqrt{5}}$ 1

So relativistic energy is given by $E = mc^2 = \frac{3m_0 c^2}{\sqrt{5}}$

Example: A space ship is moving with respect to earth with speed $\frac{\sqrt{3}}{2}c$ in positive $\hat{x}$ direction.

A particle of rest mass m_0 moving with respect to earth speed $\frac{\sqrt{3}}{2}c$ in positive $\hat{y}$ direction.



(a) Find the mass of particle with respect to observer attached to earth.

(b) Find the mass of particle with respect to observer.

Solution: (a) If observer is attached to earth, he will measure that speed of particle is $v_{P,E} = \frac{\sqrt{3}}{2}c$

so relativistic mass $m = \frac{m_0}{\sqrt{1 - \frac{|v_{P,E}|^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{3}{4}}} = 2m_0$

(b) If observer is attached to spaceship then we need to find speed of particle with respect to spaceship i.e. $v_{P,S} = ?$

It is given $v_{S,E} = \frac{\sqrt{3}}{2}c\,\hat{x} \quad v_{P,E} = \frac{\sqrt{3}}{2}c\,\hat{y}$

We assume space ship is S frame, so earth will be treated as S' $v_{S',S} = v = -\frac{\sqrt{3}}{2}c\,\hat{x}$

So $\vec{v}_{P,E} = \vec{v}_{P,S'} = u'_x = 0, u'_y = \frac{\sqrt{3}}{2}c, u'_z = 0$, $\vec{v}_{P,S} = ?$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x . v}{c^2}} = -\frac{\sqrt{3}}{2}c, \; u_y = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x . v}{c^2}} = \frac{\sqrt{3}}{2}c\sqrt{1 - \frac{3}{4}} = \frac{\sqrt{3}}{4}c, u'_z = 0$$

$$\vec{v}_{P,S} = u_x\,\hat{x} + u_y\hat{y} + u_z\hat{z} = -\frac{\sqrt{3}}{2}c\,\hat{x} + \frac{\sqrt{3}}{4}c\,\hat{y}, \left|\vec{v}_{P,S}\right| = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{\frac{3}{4} + \frac{3}{16}}\,c = \sqrt{\frac{15}{16}}\,c$$

$$m = \frac{m_0}{\sqrt{1 - \frac{\left|v_{P,S}\right|^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{15}{16}}} = 4m_0$$

Example: if kinetic energy of particle is equal to rest mass energy of particle, then find the speed of particle.

Solution: Let us assume the rest mass energy of particle is m_0c^2 and relativistic energy is

$$mc^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Kinetic energy $T = mc^2 - m_0c^2$

$$\Rightarrow m_0c^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0c^2 \Rightarrow 2 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{4} \Rightarrow v = \frac{\sqrt{3}}{2}c$$

Example: A π-meson at rest decays in to μ meson mass and neutrino ν.

If rest mass of π meson, μ meson and neutrino are m_π, m_μ and m_ν then find the total relativistic energy of outgoing μ meson and neutrino.

Solution: It is given rest mass of π meson, μ meson and neutrino are m_π, m_μ and m_ν.

Let us assume E_μ and E_ν are energies of outgoing μ meson and neutrino

$$E_\pi = m_\pi c^2 \;,\; E_\mu^2 = p_\mu^2c^2 + m_\mu^2c^4 \Rightarrow E_\mu^2 = p^2c^2 + m_\mu^2c^4$$

$$E_\nu^2 = p_\nu^2c^2 + m_\nu^2c^4 \Rightarrow E_\nu^2 = p^2c^2 + m_\nu^2c^4$$

From conservation of momentum $\vec{P}_\nu = -\vec{P}_\mu \Rightarrow |P_\nu| = |P_\mu| = P$

From conservation of energy $E_\pi = E_\mu + E_\nu \Rightarrow E_\mu + E_\nu = m_\pi c^2$

Using $E_\mu^2 - E_\nu^2 = \left(m_\mu^2 - m_\nu^2\right)c^4 \Rightarrow \left(E_\mu - E_\nu\right)\left(E_\mu + E_\nu\right) = \left(m_\mu^2 - m_\nu^2\right)c^4 \Rightarrow E_\mu - E_\nu = \frac{\left(m_\mu^2 - m_\nu^2\right)c^4}{m_\pi c^2}$

Solving equation $E_\mu - E_\nu = \frac{\left(m_\mu^2 - m_\nu^2\right)c^2}{m_\pi}$ and $E_\mu + E_\nu = m_\pi c^2$

$$E_\mu = \frac{1}{2m_\pi}\left[m_\pi^2 + m_\mu^2 - m_\nu^2\right]c^2 \quad E_\nu = \frac{1}{2m_\pi}\left[m_\pi^2 - m_\mu^2 + m_\nu^2\right]c^2$$

Example: A π-meson at rest decays in to μ meson mass and neutrino ν.

If rest mass of π meson, μ meson are m_π, m_μ and neutrino is mass less.

(a) Find the total relativistic energy of outgoing μ meson and neutrino.

(b) Find the magnitude of momentum of outgoing μ meson and neutrino.

(c) Find the speed of outgoing μ meson and neutrino.

Solution: (a) It is given rest mass of π meson, μ meson and are m_π, m_μ and neutrino is massless.

Let us assume E_μ and E_ν are energies of outgoing μ meson and neutrino

$$E_\pi = m_\pi c^2 \;,\; E_\mu^2 = p_\mu^2 c^2 + m_\mu^2 c^4 \Rightarrow E_\mu^2 = p^2 c^2 + m_\mu^2 c^4$$

$$E_\nu = p_\nu c \Rightarrow E_\nu = pc$$

From conservation of momentum $\vec{P}_\nu = -\vec{P}_\mu \Rightarrow \left|P_\nu\right| = \left|P_\mu\right| = P$

From conservation of energy $E_\pi = E_\mu + E_\nu \Rightarrow E_\mu + E_\nu = m_\pi c^2$

Using $E_\mu^2 - E_\nu^2 = m_\mu^2 c^4 \Rightarrow \left(E_\mu - E_\nu\right)\left(E_\mu + E_\nu\right) = m_\mu^2 c^4 \Rightarrow E_\mu - E_\nu = \frac{m_\mu^2 c^4}{m_\pi c^2}$

Solving equation $E_\mu - E_\nu = \frac{m_\mu^2 c^2}{m_\pi}$ and $E_\mu + E_\nu = m_\pi c^2$

$$E_\mu = \frac{1}{2m_\pi}\left[m_\pi^2 + m_\mu^2\right]c^2 \text{ and } E_\nu = \frac{1}{2m_\pi}\left[m_\pi^2 - m_\mu^2\right]c^2$$

(b) $E_\nu = p_\nu c \Rightarrow E_\nu = pc$ and $\left|P_\nu\right| = \left|P_\mu\right| = P$

$$E_\nu = \frac{1}{2m_\pi}\left[m_\pi^2 - m_\mu^2\right]c^2 = p_\nu c = \frac{1}{2m_\pi}\left[m_\pi^2 - m_\mu^2\right]c^2 \Rightarrow p_\nu = p_\mu = p = \frac{1}{2m_\pi}\left[m_\pi^2 - m_\mu^2\right]c$$

(c) Hence the neutrino is massless so it will be moving with speed of light i.e. $v_\nu = c$

Now we will calculate the speed of μ meson we know

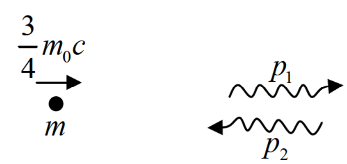
$$E_\mu = \frac{m_\mu c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\left(m_\pi^2 + m_\mu^2\right)c^2}{2m_\pi} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{2m_\pi m_\mu}{\left(m_\pi^2 + m_\mu^2\right)}$$

$$\frac{v^2}{c^2} = 1-\left(\frac{2m_\pi m_\mu}{m_\pi^2 + m_\mu^2}\right)^2 \Rightarrow \frac{\left(m_\pi^2 + m_\mu^2\right)^2 - 4m_\pi^2 m_\mu^2}{\left(m_\pi^2 + m_\mu^2\right)^2} \Rightarrow \frac{\left(m_\pi^2 - m_\mu^2\right)^2}{\left(m_\pi^2 + m_\mu^2\right)^2} = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)^2$$

Example: A pion of rest mass m_0 moving with momentum $\frac{3}{4}m_0 c$ completely decays into two photons .one photon will move in direction of pion and another is moving exactly opposite to pion.

(a) Find the momentum of outgoing photons.

(b) Find the energy of outgoing photons.



Solution: (a) The momentum of pion is $p = \frac{3}{4}m_0 c$.

The energy of pion is $E_\pi = \sqrt{p^2c^2 + m_0^2c^4} = m_0c^2\left(\sqrt{\frac{9}{16}+1}\right) = \frac{5}{4}m_0c^2$

Let us assume p_1 and p_2 are momentum of outgoing photons.

The energy of outgoing photon $E_1 = p_1 c$ and $E_1 = p_2 c$.

From conservation of momentum $p_1 - p_2 = \frac{3}{4}m_0 c$.

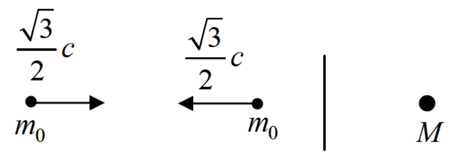
From conservation of energy $E_1 + E_2 = p_1c + p_2c = E_\pi = p_1c + p_2c = \frac{5}{4}m_0c^2$

Solving equation $p_1 - p_2 = \frac{3}{4}m_0 c$ and $p_1 + p_2 = \frac{5}{4}m_0 c$

We get $p_1 = m_0 c$ and $p_2 = \frac{m_0 c}{4}$

(b) The energy of outgoing photons are $E_1 = p_1 c = m_0c^2$ and $E_2 = p_2 c = \frac{m_0c^2}{4}$

Example: Two particles of same rest mass m_0 moving with same speed with $\frac{\sqrt{3}}{2}c$ in opposite direction. They collide head on and stick together and make composite lump. Find the mass of composite lump.



Solution: Hence total momentum before Collision is zero then from conservation of momentum composite lump will remain rest with respect to observer. Rest mass of composite lump is M.

From conservation of energy $\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}=Mc^2$ where $v=\frac{\sqrt{3}}{2}c$

$$\frac{m_0c^2}{\sqrt{1-\frac{3}{4}}}+\frac{m_0c^2}{\sqrt{1-\frac{3}{4}}}=Mc^2 \Rightarrow Mc^2=2m_0c^2+2m_0c^2 \Rightarrow M=4m_0$$

Example: A heavy particle of rest mass M while moving along the positive z - direction, decays into two identical light particles with rest mass m (where $M>2m$). Find the maximum value of the momentum that any one of the lighter particles can have in a direction perpendicular to the z - direction.

Solution: Let P be the momentum of heavy mass M, P_1 be the momentum of the light particles of mass m in the direction perpendicular to z and P_2 be the momentum in z -direction. According to conservation of momentum,

Momentum of mass M, $P=P_2+P_2=2P_2 \Rightarrow P_2=P/2$, Energy of mass M, $E=\sqrt{P^2c^2+M^2c^4}$

Momentum of a mass $m=\sqrt{P_1^2+P_2^2}=\sqrt{P_1^2+\frac{P^2}{4}}$

Energy of mass m, $E_1^2=\left(P_1^2+\frac{P^2}{4}\right)c^2+m^2c^4$

As energy is conserved $E=E_1+E_2=2E_1 \Rightarrow E_1=\frac{E}{2}$ $\qquad \because E_1=E_2$

Thus $E_1^2=\frac{E^2}{4}=\left(P_1^2+\frac{P^2}{4}\right)c^2+m^2c^4 \qquad \Rightarrow 4\left(P_1^2+\frac{P^2}{4}\right)c^2+4m^2c^4=P^2c^2+M^2c^4$

$$4P_1^2c^2+P^2c^2+4m^2c^4=P^2c^2+M^2c^4 \Rightarrow 4P_1^2c^2+4m^2c^4=M^2c^4 \Rightarrow 4P_1^2=M^2c^2-4m^2c^2$$

$$P_1^2=\frac{c^2}{4}\left(M^2-4m^2\right) \Rightarrow P_1=\frac{c}{2}\sqrt{M^2-4m^2}$$