Chapter One

LORENTZ TRANSFORMATION

1.2 Michelson Morley and Lorentz Transformation

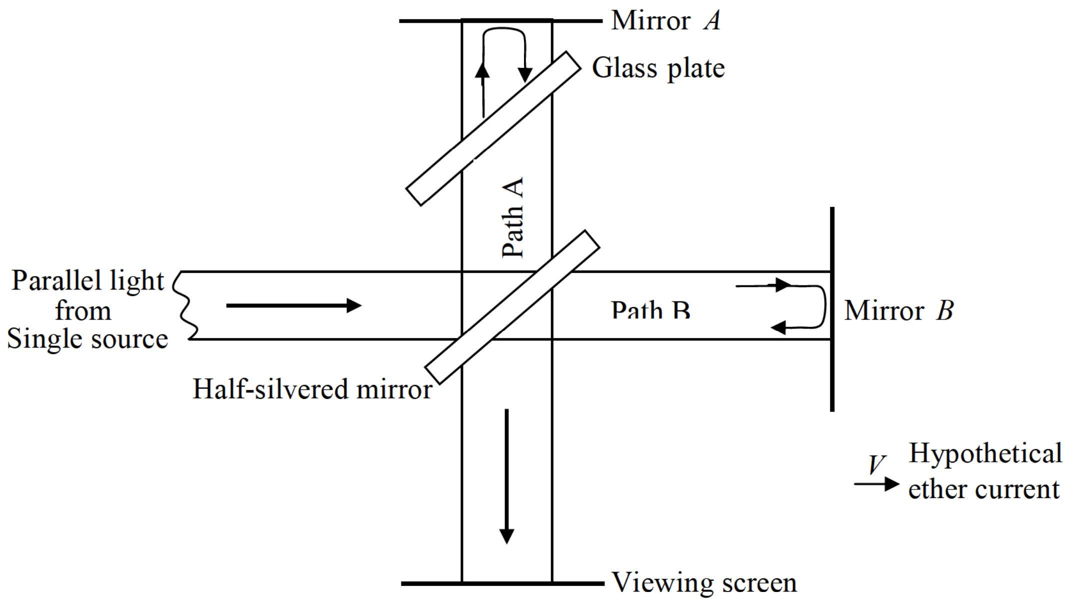


Figure 1.2

Albert A. Michelson (1852–1931) was born in Germany but came to the United States at the age of two with his parents, who settled in Nevada. He attended the U.S. Naval academy at Annapolis where, after two years of sea duty, he became a science instructor.

To improve his knowledge of optics, in which he wanted to specialize, Michelson went to Europe and studied in Berlin and Paris. Then he left the Navy to work first at the Case School of Applied Science in Ohio, then at Clark University in Massachusetts, and finally at the University of Chicago, where he headed the physics department from 1892 to 1929. Michelson's specialty was high-precision measurement, and for many decades his successive figures for the speed of light were the best available. He redefined the meter in terms of wavelengths of a particular spectral line and devised an interferometer that could determine the diameter of a star (stars appear as points of light in even the most powerful telescopes). Michelson's most significant achievement, carried out in 1887 in collaboration with Edward Morley, was an experiment to measure the motion of the earth through the "ether," a hypothetical medium pervading the universe in which light waves were supposed to occur. The notion of the ether was a hang-over from the days before light waves were recognized as electromagnetic, but nobody at the time seemed willing to discard the idea that light propagates relative to some sort of universal frame of reference.

To look for the earth's motion through the ether, Michelson and Morley used a pair of light beams formed by a half-silvered mirror, as in figure 1.2. One light beam is directed to a mirror along a path perpendicular to the ether current, and the other goes to a mirror along a path parallel to the ether current. Both beams end up at the same viewing screen. The clear glass plate ensures that both beams pass through the same thicknesses of air and glass. If the transit times of the two beams are the same, they will arrive at the screen in phase and will interfere constructively. An ether current due to the earth's motion parallel to one of the beams, however, would cause the beams to have different transit times and the result would be destructive interference at the screen. This is the essence of the experiment.

Although the experiment was sensitive enough to detect the expected ether drift, to everyone's surprise none was found. The negative result had two consequences. First, it showed that the ether does not exist and so there is no such thing as "absolute motion" relative to the ether: all motion is relative to a specified frame of reference, not to a universal one. Second, the result showed that the speed of light is the same for all observers, which is not true of waves that need a material medium in which to occur (such as sound and water waves). The Michelson-

Morley experiment set the stage for Einstein's 1905 special theory of relativity, a theory that Michelson him-self was reluctant to accept. Indeed, not long before the flowering of relativity and quantum theory revolutionized physics, Michelson announced that "physical discoveries in the future are a matter of the sixth decimal place." This was a common opinion of the time. Michelson received a Nobel Prize in 1907, the first American to do so.

From Michelson Morley Experiment it is concluded that speed of light is invariant under any Inertial frame. So, there is Discrepancy that Galilean transformation is not adequate for any speed which will approach speed of light.

Lorentz Transformation

Postulates of Special Theory of Relativity

(a) There is no universal frame of reference pervading all of space, so there is no such thing as "absolute motion".

(b) The law of physics is the same in all frames of reference moving at constant velocity.

(c) The speed of light in free space has the same value for all inertial observers. Any matter can not achieve speed of light i.e., $c = 3\times10^8 m/\sec$.

Hypothesis of Lorentz Transformation

(a) It is linear in x and x', so that a single event in frame S corresponds to a single event in frame S'.

(b) For lower velocity it reduces to Galilean transformation.

(c) If Lorentz transformation exist, then inverse transformation must exist by changing the sign of frame.

Derivation of Lorentz Transformation

Let us assume there is point in space A having space coordinates x, y, z and time coordinate t from inertial frame S. If same point A is measured from S' frame then space coordinates are x', y', z' and time t'.

According to hypothesis $x' = \gamma(x - vt)$

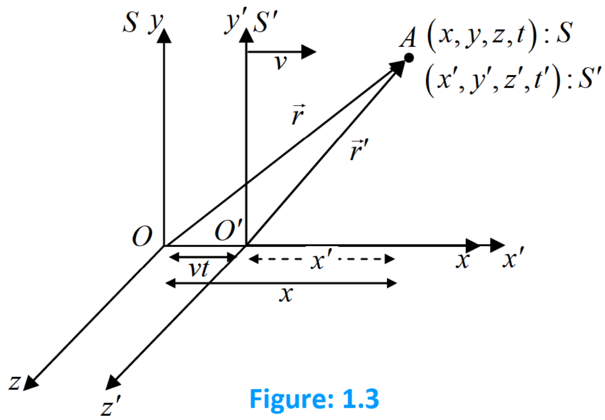


Figure: 1.3

Hence inverse Lorentz must exist then $x = \gamma(x' + vt')$, $y' = y$ and $z' = z$. The value of γ is same in both reference frame since only difference between S and S' is change of sign of velocity of reference frame v.

Put the value of $x' = \gamma(x - vt)$ in $x = \gamma(x' + vt')$, one will get $x = \gamma^2(x - vt) + \gamma vt'$

and $t' = \gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right)x$.

It is assumed that at $t = t' = 0$ the origin of both S and S' is same. We assume that light pulse emerged at $t = t' = 0$ and moves in $\hat{i}$ direction.

From S frame the relation between position and time is $x = ct$ in S frame and $x' = ct'$ in S' frame.

Use relation $x' = ct'$ put the value of x' and t' as function of x and t.

Solve in another way

Method 1: Multiply $x' = \gamma(x - vt)$ to $x = \gamma(x' + vt')$

$$x'x = \gamma^2(x - vt)(x' + vt') \Rightarrow x'x = \gamma^2(xx' - vxt' + vtx' - v^2t't)$$

Divide both side by xx' so we get

$$1 = \gamma^2\left(1 - \frac{vt}{x} + \frac{vt'}{x'} - v^2\frac{t}{x}.\frac{t'}{x'}\right) \Rightarrow 1 = \gamma^2\left(1 - \frac{v}{c} + \frac{v}{c} - \frac{v^2}{c^2}\right) \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Method 2: From S frame the relation between position and time is $x = ct$ in S frame and $x' = ct'$ in S' frame.

Use relation $x' = ct'$ put the value of x' and t' as function of x and t.

Solve in another way

$$\gamma(x-vt)=c\gamma t+\left(\frac{1-\gamma^2}{\gamma v}\right)cx$$

$$x\left(\gamma-\left(\frac{1-\gamma^2}{\gamma v}\right)c\right)=ct\left(\gamma+\frac{\gamma v}{c}\right)\Rightarrow\ x=ct\frac{\gamma+\frac{\gamma v}{c}}{\gamma-\left(\frac{1-\gamma^2}{\gamma v}\right)c}$$

Compare above equation with $x=ct$ we get

$$\frac{\gamma+\frac{\gamma v}{c}}{\gamma-\left(\frac{1-\gamma^2}{\gamma v}\right)c}=1\Rightarrow\gamma+\frac{\gamma v}{c}=\gamma-\left(\frac{1-\gamma^2}{\gamma v}\right)c\Rightarrow\frac{\gamma^2v^2}{c^2}=\gamma^2-1\Rightarrow\gamma^2=1-\frac{v^2}{c^2}\ \Rightarrow\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Put the value of $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ in equation $x'=\gamma(x-vt)$ and $t'=\gamma t+\left(\frac{1-\gamma^2}{\gamma v}\right)$ we get

Lorentz transformation and inverse Lorentz transformation as

Lorentz transformation	**Inverse Lorentz transformation**
$x'=\gamma(x-vt)=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$ $y'=y$ $z'=z$ $t'=\gamma\left(t-\frac{vx}{c^2}\right)=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	$x=\gamma(x'+vt')=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$ $y=y'$ $z=z'$ $t=\gamma\left(t'+\frac{vx'}{c^2}\right)=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$

Example: Two events in S frame is separated by space coordinate $x_2 - x_1 = L$ and by time coordinate $t_2 - t_1 = T$.

(a) What will be difference in space coordinate measured from S' which is moving with speed v with respect to S frame.

(b) What will be difference in time coordinate measured from S' which is moving with speed v with respect to S frame.

Solution: From S frame $x_2 - x_1 = L$ and $t_2 - t_1 = T$ from S' frame $x_2' - x_1' = ?$ and $t_2' - t_1' = ?$

(a) $$L' = x_2' - x_2' = \left(\frac{(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}}\right) - \left[\frac{v(t_2 - t_1)}{\sqrt{1 - \frac{v^2}{c^2}}}\right] = \frac{L - vT}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(b) $$T' = t_2' - t_1' = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_2 - t_1}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{v}{c^2}\frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{v}{c^2}\frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$

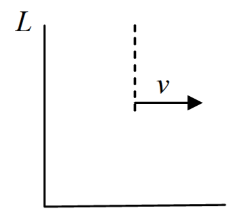
Example: A relativistic particle moves with a constant velocity v with respect to the laboratory frame. In time τ, measured in the rest frame of the particle,

(a) What will distance travelled by the particle in its own reference frame.

(b) If τ is time measured by particle in particle frame what is time measured in laboratory frame.

(c) What is the distance that it travels in the laboratory frame .

Solution: Assume laboratory is S frame, the particle is moving with respect to laboratory so it is S' frame.



(a) From Particle frame of reference

$$x_1' = 0, x_2' = 0 \quad t_{initial} = t_1', t_{final} = t_2' \Rightarrow t_2' - t_1' = \tau$$

$$d = x_2' - x_1' = 0$$

(b) $$t_2 - t_1 = \frac{t_2' - \frac{vx_2'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1' - \frac{vx_1'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{v}{c^2}\left(\frac{x_2' - x_1'}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(c) $$x_1 = \frac{x_1' + vt_1'}{\sqrt{1 - v^2/c^2}}, \; x_2 = \frac{x_2' + vt_2'}{\sqrt{1 - v^2/c^2}}$$

$$x_2 - x_1 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} + \frac{v(t_2' - t_1')}{\sqrt{1 - v^2/c^2}} \Rightarrow \Delta x = \frac{v(t_2' - t_1')}{\sqrt{1 - v^2/c^2}} = \frac{v\tau}{\sqrt{1 - v^2/c^2}}$$