Chapter One

LORENTZ TRANSFORMATION

1.4 Consequences of Lorentz Transformation (Length Contraction, Time Dilation and Loss of Simultaneity)

(a) Length Contraction:

Moving rod appeared shorter

Proper Length: If relative speed between length of rod and observer is zero then observer will measure proper length.

Contracted Length: If there is relative speed between length of rod and observer then observer will measure contracted length which is shorter than proper length.

Suppose a rod of proper length l_0 lying along $\hat{x}$ axis of S' frame which is moving with speed v with respect to S along $\hat{x}$ direction.

In order to measure the length of an object in motion, relation to observer, the position of two end points recorded simultaneously, the length of object in direction of motion appeared smaller to observer.

If observer is attached to S' he will measure proper length i.e. $l_0 = x'_2 - x'_1$

If observer is attached to S he will measure proper length i.e. $l = x_2 - x_1$

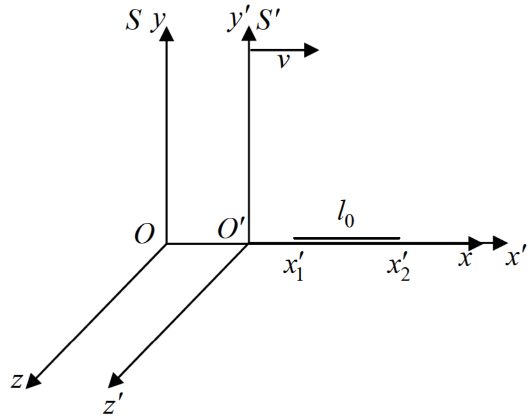


Figure 1.5

From Lorentz transformation

$$x'_2 - x'_1 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1)$$

$$(x'_2 - x'_1) = \gamma(x_2 - x_1) \text{ because } t_2 = t_1$$

$$(x'_2 - x'_1) = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} \Rightarrow (x_2 - x_1) = \sqrt{1 - \frac{v^2}{c^2}} \quad (x'_2 - x'_1)$$

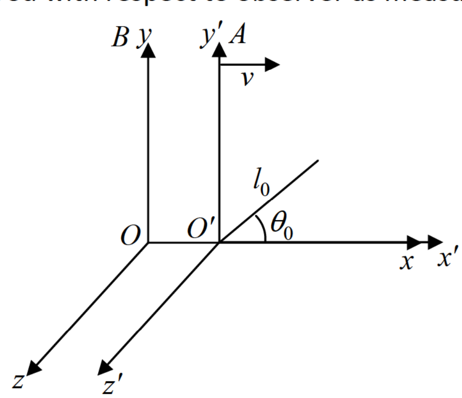
$$\Rightarrow l = l_o\sqrt{1 - \frac{v^2}{c^2}} \quad \text{so } l < l_0$$

$l < l_0$, which means moving rod appeared shorter, such a contraction of length in direction of motion relative to observer is called Lorentz Fitzgerald contradiction.

Example: A rod of proper length l_0 making angle θ_0 with x-axis of frame A. If frame A is moving with respect observer along x- axis.

(a) Determine the length of rod measured by observer.

(b) What will angle made by rod with respect to observer as measured by observer.



Solution: Proper length of the rod in the direction of moving frame, $l_{x_0} = l_0 \cos\theta_0$, and perpendicular to motion is $l_{y_0} = l_0 \sin\theta_0$

(a) Length measured in the moving frame $l_x = l_0 \cos\theta_0 \sqrt{1 - \frac{v^2}{c^2}}$ and $l_y = l_0 \sin\theta_0$

$$|l| = \sqrt{l_0^2\cos^2\theta_0\left(1-\frac{v^2}{c^2}\right)+l_0^2\sin^2\theta_0} = l_0\sqrt{\cos^2\theta_0\left(1-\frac{v^2}{c^2}\right)+\sin^2\theta_0} = l_0\sqrt{1-\left(\frac{v\cos\theta_0}{c}\right)^2}$$

(b) $\tan\theta' = \dfrac{l_y}{l_x} = \dfrac{l_0\sin\theta_0}{l_0\cos\theta_0\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \tan\theta' = \gamma\tan\theta_0,\ \theta' = \tan^{-1}(\gamma\tan\theta_0)$

Example: The area of a disc in its rest frame S is equal to 1 (in some units). The disc will appear distorted to an observer O moving with a speed u with respect to S along the plane of the disc. Find the area of the disc measured in the rest frame of the observer O is (c is the speed of light in vacuum)

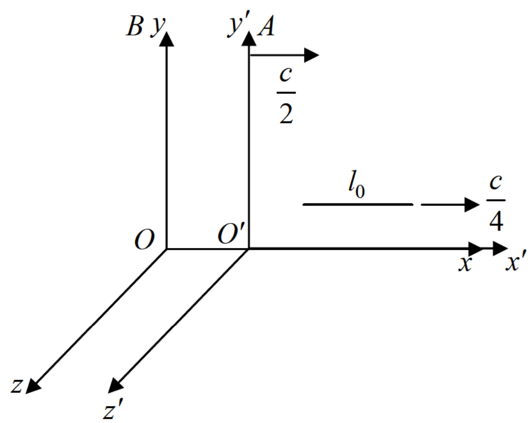
Solution: Area of disc from S - frame is 1 i.e., $\pi a^2 = 1$ or $\pi a\cdot a = 1$

Due to length contraction, the circle will appeared as ellipse with major axis a and minor axis $b = a\sqrt{1-\frac{u^2}{c^2}}$

Area of disc from S' frame is $\pi a\cdot b = \pi a\cdot a\sqrt{1-\frac{u^2}{c^2}} = 1\cdot\sqrt{1-\frac{u^2}{c^2}} = \sqrt{1-\frac{u^2}{c^2}}$

where u is the speed of circle with respect to observer.

Example: A rod of proper length l_0 moving with respect to A with speed velocity $\frac{c}{4}\hat{i}$. The frame A is moving with respect to B with velocity $\frac{c}{4}\hat{i}$



(a) Find length of rod measured by observer attached to rest frame of A

(b) Find length of rod measured by observer attached to rest frame of B

Solution: (a) velocity of rod with respect to A $v_{R,A} = \frac{c}{4}$

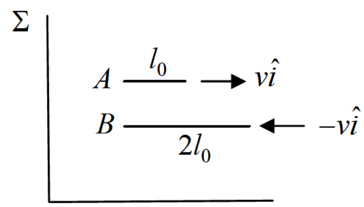
$$l = l_0\sqrt{1-\frac{(v_{R,A})^2}{c^2}} = l_0\sqrt{1-\frac{1}{16}} = \sqrt{\frac{15}{16}}l_0$$

(b) velocity of rod with respect to A $v_{R,A} = v_{S',A} = u'_x = \frac{c}{4}$

velocity of $v_{A,B}=v_{S',S}=v=\frac{c}{2}$ $v_{R,B}=v_{R,S}=u_x=\frac{u_x'+v}{1+\frac{u_x'v}{c^2}}=\frac{\frac{c}{2}+\frac{c}{4}}{1+\frac{\frac{c}{2}\cdot\frac{c}{4}}{c^2}}$

$$l=l_0\sqrt{1-\frac{\left(v_{R,B}\right)^2}{c^2}}=l_0\sqrt{1-\frac{4}{9}}=\sqrt{\frac{5}{9}}l_0$$

Example: Two rod A and B moving with respect to common frame Σ with velocity $v\hat{i}$ and $-v\hat{i}$ respectively. The proper length of A is l_0 and proper length of B is $2l_0$. Find the value of v such that length of rod B will appeared l_0 from the observer is attached to rest frame of A.



Solution: $v_{A,\Sigma}=v\hat{i},\ v_{B,\Sigma}=-v\hat{i}$

First, we need to calculate $v_{B,A}$

$$A\equiv S,\Sigma\equiv S'\quad v_{S',S}=-v\hat{i}\quad v_{B,\Sigma}=v_{B,S'}=u_x'=-vi$$

$$v_{B,A}=v_{B,S}=u_x\quad u_x=\frac{u_x'+v}{1+\frac{u_x'v}{c^2}}=\frac{-v-v}{1+\frac{(-v)(-v)}{c^2}}=\frac{-2v}{1+\frac{v^2}{c^2}}$$

$$l_0=2l_0\sqrt{1-\frac{u_x^2}{c^2}}=l_0=2l_0=\frac{3}{4}=\frac{u_x^2}{c^2}\Rightarrow\frac{3}{4}=\frac{4\frac{v^2}{c^2}}{\left(1+\frac{v^2}{c^2}\right)^2}$$

put $\frac{v^2}{c^2}=\alpha<1$

$$\frac{3}{4}=\frac{4\alpha}{(1+\alpha)^2}\Rightarrow 16\alpha=3+3\alpha^2+6\alpha\Rightarrow 3\alpha^2-10\alpha+3=0\Rightarrow 3\alpha(\alpha-3)-1(\alpha-3)=0$$

So $\alpha=\frac{1}{3},\alpha=3$ the accepted solution is so $v=\frac{c}{\sqrt{3}}$