# Chapter One
# Superposition Principle

We use the term interference in our daily life. Can someone describe its meaning? In layman terms, we describe interference as someone intruding our space beyond a certain limit. So, what happens when someone starts interfering in your life? Naturally, there will be a time when there will be disagreements between the two people say on taking some important decision. We say that they are going through a dark phase in their lives. Then there will also be a time when both will agree on some decision, this can be regarded as a bright phase in their lives. This is a scenario that we typically observe in our day-to-day lives. Now, what is interesting is that a similar phenomenon is also observed in case of light as well. That is when two light beams are brought together such that there's is some overlap between the two then we observe a region of alternate dark and bright phase, known as interference pattern. In other words we can say that superposition of two light beams is called interference. Just like you cannot have arguments with an unknown person, there has to be some relation between the people. Similarly, for the light beams to interfere there has to be a fixed relation between the phase of light waves of same frequency. Such light sources are said to be in coherence. This is possible only when both the light waves are generated from a single source. So, if one superimpose two light beams coming from two different source of light then you will not obtain any interference pattern. Are there any other examples of interference seen around you?
This was just a basic introduction to interference of light. Now, we all know that a light wave in mathematical terms is represented by its Electric field at a particular given as:

$$E_1 = E_{01}\cos\left(\omega t + \theta_1\right)\hat{x}$$

Another light wave of same frequency will be given by:

$$E_2 = E_{02}\cos(\omega t + \theta_2)\hat{x} \tag{1.2}$$

So, there superposition can now be written as:

$$\begin{aligned} E &= E_1 + E_2 \\ &= E_{01}\cos(\omega t + \theta_1) + E_{02}\cos(\omega t + \theta_2)\hat{x} \end{aligned} \tag{1.3}$$

Expanding the above expression using the identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$, the resultant disturbance will also be simple harmonic of the form:

$$E = E_0\cos(\omega t + \theta) \tag{1.4}$$

where,

$$E_0\cos\theta = E_{01}\cos\theta_1 + E_{02}\cos\theta_2 \tag{1.5}$$

and

$$E_0\sin\theta = E_{01}\sin\theta_1 + E_{02}\sin\theta_2 \tag{1.6}$$

Now, to obtain $E_0$ we simply have to square and add eqns. (1.5) and (1.6) to obtain:

$$E_0 = \left[E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\theta_1 - \theta_2)\right]^{1/2} \tag{1.7}$$

Further $\theta$ can be written in the form:

$$\tan\theta = \frac{E_{01}\sin\theta_1 + E_{02}\sin\theta_2}{E_{01}\cos\theta_1 + E_{02}\cos\theta_2} \tag{1.8}$$

In equation 1.7, if $\theta_1 \sim \theta_2 = 0, 2\pi, 4\pi, ...$, then

$$E_0 = E_{01} + E_{02} \tag{1.9}$$

and what we obtain is a bright field. Thus, if two light waves are in phase, then the resultant wave amplitude will be the sum of the two amplitudes to obtain a bright region. This is also known as *constructive interference*. Similarly, if $\theta_1\ \theta_2 = \pi, 3\pi, 5\pi, ...$, then

$$E_0 = E_{01} - E_{02} \tag{1.10}$$

Thus, we obtain a dark region, also known as *destructive interference*, as the resultant wave amplitude is the difference of the two amplitudes.

We also know that the intensity of light at a given point is given by the square of its electric field. Therefore, the net interference intensity at a point can be given as:

$$I = |E|^2 = |E_1 + E_2|^2 \tag{1.11}$$

Now an important question comes, that is, when this constructive and destructive interference is happening then is the law of conservation of energy violated? Well, the principle of energy conservation is not getting violated; but what actually is happening is that the energy is just being redistributed.

For a general case, with n waves given by:

$$\begin{aligned} E_1 &= E_{01}\cos(\omega t + \theta_1)\hat{x} \\ E_2 &= E_{02}\cos(\omega t + \theta_2)\hat{x} \\ &\dots\dots\dots\dots\dots\dots\dots \\ E_n &= E_{0n}\cos(\omega t + \theta_n)\hat{x} \end{aligned} \tag{1.12}$$

The resultant wave can be written as

$$E = E_1 + E_2 + \dots + E_n = E_0\cos(\omega t + \theta), \tag{1.13}$$

where,

$$E_0\cos\theta = E_{01}\cos\theta_1 + \dots + E_{0n}\cos\theta_n \tag{1.14}$$

and,

$$E_0\sin\theta = E_{01}\sin\theta_1 + \dots + E_{0n}\sin\theta_n \tag{1.15}$$

Let us now consider the superposition of two orthogonally polarised light waves, i.e. two light waves whose electric field vectors are oscillating in perpendicular direction to each other. Let's begin by taking the following two equations:

$$\begin{aligned} E_x &= E_1\cos(\omega t)\hat{x} \\ E_y &= E_2\cos(\omega t + \theta)\hat{y} \end{aligned} \tag{1.16}$$

Let us first solve these two equation by **Analytical Method**. We will consider a few cases for this.

**Case I:** When $\theta = 0 \; or \; \pi$

$$E_x = E_1\cos(\omega t)$$
$$E_y = E_2\cos(\omega t)$$

$$i.e \qquad E_y = -\frac{E_2}{E_1}E_x \tag{1.17}$$

Similarly, when $\theta = \pi$

$$E_x = E_1 \cos(\omega t)$$
$$E_y = -E_2 \cos(\omega t)$$

$$i.e \qquad E_y = -\frac{E_2}{E_1}E_x \tag{1.18}$$

Thus, from eqn (17) & (18) we find that the resultant is a straight line passing through origin. For $\theta = 0$, the motion is along the OB and for $\theta = \pi$ the motion is along OE as shown if the following Fig. 1.1.

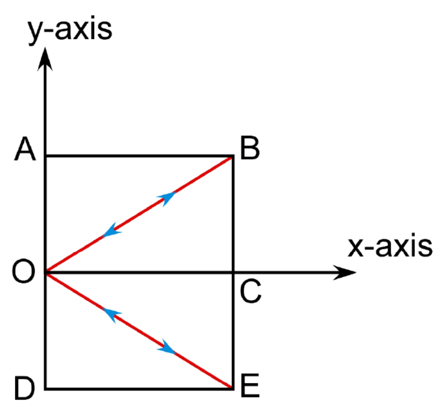


Fig. 1.1

**Case II:** For $\theta = \pi/2$

In this case the two fields will be given as

$$E_x = E_1 \cos(\omega t)$$
$$E_y = E_2 \cos(\omega t + \pi/2) = -E_2 \sin(\omega t)$$

Squaring and adding both these equations, we get

$$\frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_2^2} = \cos^2(\omega t) + \sin^2(\omega t) = 1 \tag{1.19}$$

This is the standard equation of an ellipse whose principal axes lie along the x- and y-axis. The semi major and semi-minor axes of the ellipse are $E_1$ and $E_2$ if $E_1 > E_2$. From the above

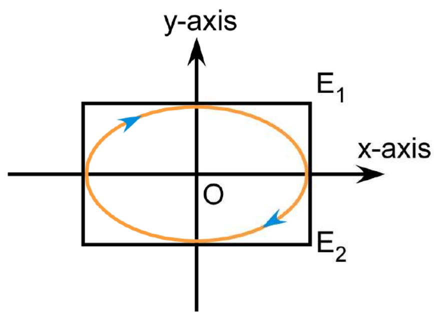


Fig. 1.2

equations, we can see that as time progresses, $E_x$ decreases from its maximum value $E_1$ and at the same time $E_y$ becomes more and more negative. Thus the ellipse will be described in clockwise direction as shown in Fig. 1.2 Similarly, for the case for $\theta = 3\pi/2$ or $\theta = -\pi/2$, we shall get the same ellipse but the motion will now be in anticlockwise direction (See Fig. 1.3).

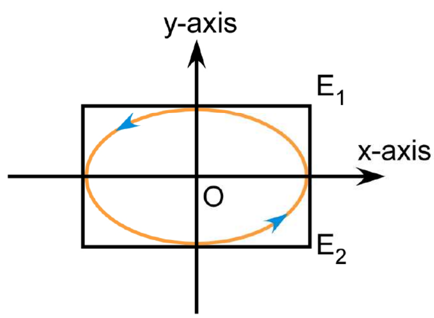


Fig. 1.3

When $E_1 = E_2 = E_0$, then equation (19) reduces to the equation of circle of radius $E_0$

$$E_x^2 + E_y^2 = E_0^2 \tag{1.20}$$

Let us now consider the general case to solve the eqn. (16). We can rewrite it as:

$$\begin{aligned} \frac{E_x}{E_1} &= \cos(\omega t) \\ \therefore \sin(\omega t) &= \sqrt{\left(1 - \frac{E_x^2}{E_1^2}\right)} \end{aligned} \tag{1.21}$$

$$\begin{aligned}\frac{E_y}{E_2} &= \cos(\omega t + \theta) = \cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)\\ &= \frac{E_x}{E_1}\cos\theta - \sqrt{\left(1-\frac{E_x^2}{E_1^2}\right)}\sin\theta \\ \sqrt{\left(1-\frac{E_x^2}{E_1^2}\right)}\sin\theta &= \frac{E_x}{E_1}\cos\theta - \frac{E_y}{E_2}\end{aligned} \tag{1.22}$$

Squaring the above equation, we get,

$$\frac{E_x^2}{E_1^2}\cos^2\theta + \frac{E_y^2}{E_2^2} - 2\frac{E_xE_y}{E_1E_2}\cos\theta = \left(1-\frac{E_x^2}{E_1^2}\right)\sin^2\theta$$

$$\frac{E_x^2}{E_1^2}\left(\cos^2\theta + \sin^2\theta\right) + \frac{E_y^2}{E_2^2} - 2\frac{E_xE_y}{E_1E_2}\cos\theta = \sin^2\theta$$

$$\frac{E_x^2}{E_1^2} + \frac{E_y^2}{E_2^2} - 2\frac{E_xE_y}{E_1E_2}\cos\theta = \sin^2\theta \tag{1.23}$$

Eq. (23) is describes an ellipse with its axes inclined to the coordinate axes. Different trajectories can be observed for different $\theta$ values. A few examples are shown in the following Fig. 1.4

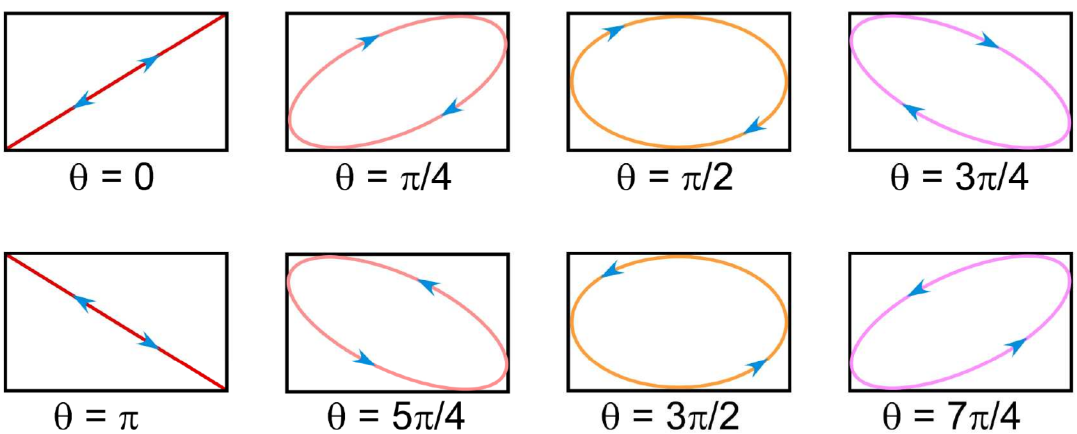


Fig. 1.4

**Conclusion for superposition of two mutually perpendicular polarized light beams:**

1. If $\theta = n\pi$, then the resultant polarization state of the superposition of the two perpendicular polarized beams will also be linearly polarized with its electric field oscillating in a direction different from both the original waves. The angle can be found by $\theta = \tan^{-1}\left(\pm\frac{E_2}{E_1}\right)$.

2. If $\theta \neq n\pi$, the resultant field vector oscillates in ellipse if $E_1 \neq E_2$ and in circle if $E_1 = E_2$

3. If $\Delta\theta = \theta_1 - \theta_2$ lies between $0 \rightarrow \pi$ then the resultant ellipse (or circle) will rotate in anticlockwise direction. Hence, for $-\Delta\theta$ it will rotate in clockwise direction as was discussed above in our case (See Fig. 1.4).

4. If $\Delta\theta$ lies between $\pi \rightarrow 2\pi$ then the resultant ellipse (or circle) will rotate in clockwise direction. Hence, for $-\Delta\theta$ it will rotate in anticlockwise direction (See Fig. 1.4).