Chapter One
Bohr Sommerfeld Theory of Quantization

1.1 Bohr Sommerfeld Theory of Quantization

For any physical system in which the coordinates are periodic function of time, there exist a quantum condition for each coordinate. These quantum conditions are

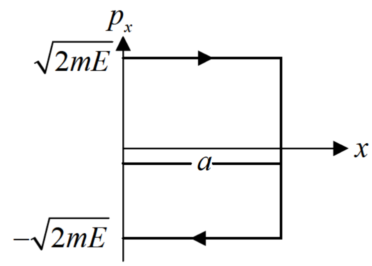
$$\oint p_q dq = n_q h$$

Where q is one of generalized coordinate p_q is the momentum associated with that coordinates, n_q is the quantum number which takes on integral values, and $\oint$ means that integration is taken over one period of the coordinate q. The h is famous plank's constant which has value of $6.6\times10^{-34}\,Joule-\sec$.

The diagram between q along horizontal axis and p_q along vertical axis are identified as phase diagram. In classical mechanics the area under the phase diagram is continuous but according to Bohr Sommerfeld theory it can be stated that area under the phase diagram is not continuous rather discrete in nature.

Example: A particle of mass m is confine in one dimensional box of width a. Consider particle is free inside box. Using Bohr Sommerfeld theory find the quantized energy of Particle.

Solution:



$$E = \frac{p_x^2}{2m}$$

$$\oint p_x dx = n_x h$$

$$\int_0^a \sqrt{2mE}dx + \int_a^0 \left(-\sqrt{2mE}\right)dx = n_x h \;, n_x = 1,2,3...$$

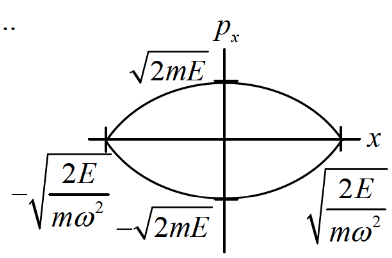
$$2\int_0^a \sqrt{2mE}dx = n_x h,\; n_x = 1,2,3.... \Rightarrow 2\sqrt{2mE}a = n_x h \Rightarrow E = \frac{n_x^2 h^2}{8ma^2} \Rightarrow \text{ where } n_x = 1,2,3...$$

One can see energy is quantized as $E_{n_x} = \dfrac{n_x^2 h^2}{8ma^2}$

Example: The classical energy of harmonic oscillator is given as $E = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$. Using Bohr Sommerfeld theory find the possible quantized energy.

Solution: $E = \dfrac{p_x^2}{2m} + \dfrac{1}{2}m\omega^2 x^2 \Rightarrow p_x = \sqrt{2m\left(E - \dfrac{1}{2}m\omega^2 x^2\right)}$

$\oint p_x dx = n_x h$ where $n_x = 1,2,3...$



$$4\int_0^{\left(\frac{2E}{m\omega^2}\right)^{1/2}} \sqrt{2m\left(E - \frac{m\omega^2 x^2}{2}\right)}dx = n_x h \Rightarrow \sqrt{2mE}.4\int_0^{\left(\frac{2E}{m\omega^2}\right)^{1/2}} \sqrt{\left(1 - \frac{m\omega^2 x^2}{2E}\right)}dx = n_x h$$

Put the substitution $\left(\dfrac{m\omega^2}{2E}\right)^{1/2} x = t \Rightarrow dx = \left(\dfrac{2E}{m\omega^2}\right)dt$ and

The limit of integration varied as $(x = 0, t = 0)$ and $\left(x = \left(\dfrac{2E}{m\omega^2}\right)^{1/2}, t = 1\right)$

The integration can be evaluated as $4.\sqrt{2mE}\sqrt{\dfrac{2E}{m\omega^2}}.\int_0^1 \sqrt{1-t^2}dt = n_x h,\; n_x = 1,2,3...$

Again put $t = \sin\theta, dt = \cos\theta d\theta$ and limit will change from 0 to $\frac{\pi}{2}$

$$4.\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}.\int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\cos\theta d\theta = n_x h, \Rightarrow 4\sqrt{2mE}.\sqrt{\frac{2E}{m\omega^2}}\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

where $\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta = \frac{\pi}{4}$

$$4\sqrt{2mE}.\sqrt{\frac{2E}{m\omega^2}}\frac{\pi}{4} = n_x h \Rightarrow \frac{2\pi E}{\omega} = n_x h \Rightarrow E = n_x\frac{h\omega}{2\pi}$$

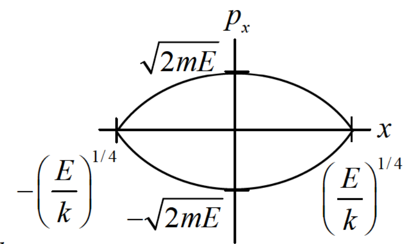
So quantized energy is $E_{n_x} = n_x\hbar\omega$ where $n_x = 1, 2, 3...$

Example: The classical energy of system is given as $E = \frac{p_x^2}{2m} + kx^4$.

(a) For given value of Energy E area under the phase diagram is proportional to $E^{3/4}$

(b) Using Bohr Sommerfeld theory the possible quantized energy proportional to $(n_x)^\alpha$ where n_x is Bohr quantization number then find the value of α.

Solution: $E = \frac{p_x^2}{2m} + kx^4 \Rightarrow p_x = \sqrt{2m\left(E - kx^4\right)}$



Where area under the phase curve is $\oint p_x dx = A$

$$A = 4\int_0^{\left(\frac{E}{k}\right)^{1/4}}\sqrt{2m\left(E - kx^4\right)}dx = n_x h \Rightarrow \sqrt{2mE}.4\int_0^{\left(\frac{E}{k}\right)^{1/4}}\sqrt{\left(1 - \frac{kx^4}{E}\right)}dx$$

Put the substitution $\left(\frac{k}{E}\right)^{\frac{1}{4}}x = t \Rightarrow dx = \left(\frac{E}{k}\right)^{\frac{1}{4}}dt$ and

The limit of integration varied as $(x = 0, t = 0)$ and $\left(x = \left(\frac{E}{k}\right)^{1/4}, t = 1\right)$

The integration can be evaluated as $A = 4.\sqrt{2mE}\left(\frac{E}{k}\right)^{1/4}\int_0^1\sqrt{1-t^4}dt$

The integration $\int_0^1\sqrt{1-t^4}dt = \beta$ where β some constant value $(0 < \beta < 1)$

So $A \propto \sqrt{2mE}\left(\frac{E}{k}\right)^{1/4} \Rightarrow A \propto E^{\left(\frac{1}{2}+\frac{1}{4}\right)} \Rightarrow A \propto E^{3/4}$

$\propto n_x \Rightarrow E \propto (n_x)^{4/3}$

(b) From Bohr Sommerfeld theory $\oint p_x dx = n_x h$ where $n_x = 1, 2, 3 \ldots$

$\oint p_x dx = A = n_x h \propto E^{3/4} \propto n_x \Rightarrow E \propto n_x^{4/3}$

So quantized value of $E_{n_x} \propto n_x^{4/3}$ so value of α is $\frac{4}{3}$