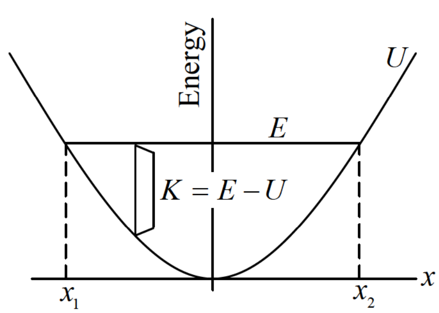
Chapter One
Stability Analysis and Phase Diagram

1.1 Energy Diagrams

We can often find the most interesting features of the motion of a one dimensional system by using an energy diagram, in which the total energy E and the potential energy U are plotted as functions of position. The kinetic energy $K = E - U$ is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where $U \leq E$.

Energy Diagram of Bounded Motion



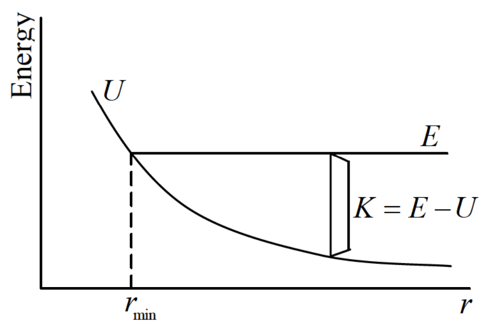
Here is the energy diagram for a harmonic oscillator. The potential energy $U = kx^2/2$ is a parabola centered at the origin. Since the total energy is constant for a conservative system, E is represented by a horizontal straight line. Motion is limited to the shaded region where $E \geq U$; the limits of the motion, x_1 and x_2 in the sketch, are sometimes called the turning points.

Here is what the diagram tells us. The kinetic energy, $K = E - U$ is greatest at the origin. As the particle flies past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points x_1, x_2. The particle then moves toward the origin with increasing kinetic energy and the cycle is repeated.

The harmonic oscillator provides a good example of bounded motion. As E increases, the turning points move farther and farther off, but the particle can never move away freely. If E decreased, the amplitude of motion decreases, until finally for $E = 0$ the particle lies at rest at $x = 0$.

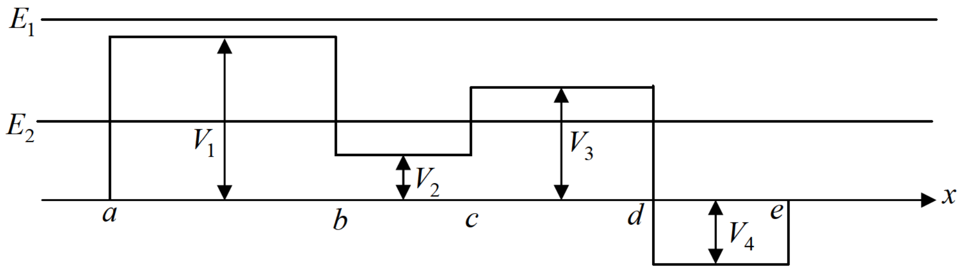
Energy Diagram of Unbounded Motion

The pot $U = A/r$, where A is positive. There is a distance of closest approach $r_{\min}$, as shown in the diagram, but the motion is not bounded for large r since U decreases with distance. If the particle is shot toward the origin, it gradually losses kinetic energy until it comes momentarily to rest at $r_{\min}$. The motion then reverses and the particle moves back towards infinity. The final and initial speeds at any point are identical; the collision merely reverses the velocity.



For positive energy, $E > 0$, the motion is unbounded, and the atoms are free to fly apart. As the diagram indicates, the distance of closest approach, $r_{\min}$, does not change appreciably as E is increased. The kinetic energy will be zero at $r_{\min}$ and as r increases the potential energy decreases, but kinetic energy $K = E - U$ will increase sharply.

Example:



In the above figure, potential energy in different regions are given, where

$$V(x)=\begin{cases} V_1 = 8J, & a<x<b \\ V_2 = 3J, & b<x<c \end{cases} \quad \text{and} \quad V(x)=\begin{cases} V_3 = 6J, & c<x<d \\ V_4 = -4J, & d<x<e \end{cases}$$

The potential is assumed to be zero in all other regions.

(a) What will be kinetic energy in all region if total energy E is $10J$?

Solution: If 'T' is kinetic energy and 'V' is potential energy, then total energy $E = T + V$, so kinetic energy is $T = E - V$.

For total energy $E = E_1$ all regions are classical allowed region.

So, in region $x < a$, $V(x) = 0$, so $T = 10 - 0 = 10J$

In region $a < x < b$, $V(x) = V_1 = 8J$ so $T = 10 - 8 = 2J$

In region $b < x < c$, $V(x) = V_2 = 3J$ so $T = 10 - 3 = 7J$

In region $c < x < d$, $V(x) = V_3 = 6J$ so $T = 10 - 6 = 4J$

In region $d < x < e$, $V(x) = V_4 = -4$ so $T = 10 - (-4) = 14J$

In region $e < x$, $V(x) = 0$ so $T = 10 - 0 = 10J$

(b) What will be kinetic energy in all regions, if total energy E is $5J$?

Solution: If T is kinetic energy and V is potential energy then total energy $E = T + V$, so kinetic energy is $T = E - V$

So, in region $x < a$, $V(x) = 0$ so, $T = 5 - 0 = 5J$

In region $a < x < b$, $V(x) = V_1 = 8J$ hence $V_1 > E_2$ so, $T = 0$ (classical forbidden region)

In region $b < x < c$, $V(x) = V_2 = 3J$ so, $T = 5 - 3 = 2J$

In region $c < x < d$, $V(x) = V_3 = 6J \Rightarrow V_3 > E_2$ so, $T = 0$ (classical forbidden region)

In region $d < x < e$, $V(x) = V_4 = -4$ so, $T = 5 - (-4) = 9J$

In region $e < x$, $V(x) = 0$ so, $T = 5 - 0 = 5J$

1.2 Stability and Instability in One Dimension

Equilibrium Point: Any potential can be function of generalized coordinate, generalized velocity and time $V \equiv V(x, \dot{x}, t)$. The equilibrium point is defined where total external force on the system is zero i.e. for any co-ordinate say x is said to be equilibrium point if $\frac{\partial V}{\partial x} = 0$ at $x = x_0$

Unstable Equilibrium Point: If x_0 is maxima or (local maxima) i.e. $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} < 0$, then it is said to be unstable equilibrium point. Unstable equilibrium point always behaves like repulsive point.

Stable Equilibrium Point: If x_0 is minima or (local minima) i.e. $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} > 0$, then it is said to be stable equilibrium point. Stable equilibrium point always behaves as an attractive point.

Example: If potential in one dimension is given by $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ then

(a) Find the point where potential is zero

(b) Find the equilibrium point.

(c) Find the stable and unstable equilibrium point

(d) Draw phase curve i.e. $V(x)$ vs x for given energy

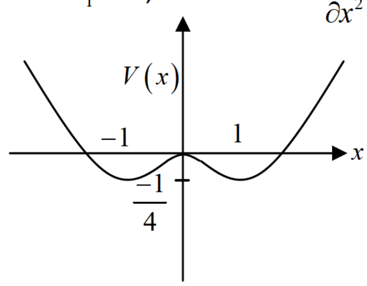
Solution: (a) $V(x) = 0 \Rightarrow -\frac{x^2}{2} + \frac{x^4}{4} = 0 \ \Rightarrow \ x = 0, +\sqrt{2}, -\sqrt{2}$

(b) For equilibrium point $\frac{\partial V}{\partial x} = 0 \Rightarrow -x + x^3 = 0$. So there are three equilibrium points.

$$x_1 = 0,\ x_2 = 1,\ x_3 = -1$$

(c) For discussion of stability and instability, we must find, $\frac{\partial^2 V}{\partial x^2} = -1 + 3x^2$. For stable equilibrium $\frac{\partial^2 V}{\partial x^2} > 0$. At $x_2 = 1$ and $x_3 = -1$ the value of $\frac{\partial^2 V}{\partial x^2} = 2$ which is greater than 0. For unstable equilibrium point. $\frac{\partial^2 V}{\partial x^2} < 0$. At $x_1 = 0$, the value of $\frac{\partial^2 V}{\partial x^2} = -1$, which is less than 0, so it is unstable point.

(d) $V(x)$ vs x



1.3 Small Oscillations

Let us assume the potential $V(x)$ has stable equilibrium point at $x = x_0$ then $\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0$ and

$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} > 0$

The Taylor expansion of $V(x)$ about $x = x_0$ is given by

$$V(x) = V(x_0) + \left.\frac{\partial V}{\partial x}\right|_{x=x_0}(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0)^2 + \text{order}(x - x_0)^3 ......$$

If term $(x - x_0)^2$ is small then higher order terms can be neglected, then potential energy is equivalent to $V(x) = V(x_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0)^2$ because $\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0$

So, force is equal to $F = -\frac{\partial V}{\partial x}$, $F = -\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0)$. Hence, $F \propto -(x - x_0)$ and the motion is small oscillation and the angular frequency is given by $\omega = \sqrt{\frac{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}}{m}}$, where m is mass of the particle. The term $k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}$ is identified as spring constant.

Example: If particle of mass m interact with potential $ax^2 + \frac{b}{x^2}$, then what will be the frequency of oscillation? (Assume oscillation is small)?

Solution: $V(x) = ax^2 + \frac{b}{x^2}$

For equilibrium point $\frac{\partial V}{\partial x} = 0$

$$2ax - \frac{2b}{x^3} = 0 \Rightarrow ax^4 - b = 0 \Rightarrow x_0 = \pm\left(\frac{b}{a}\right)^{1/4}$$

$$k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = 2a + \frac{2.3b}{x^4}$$. Now, put the value of $x_0 = \pm\left(\frac{b}{a}\right)^{1/4}$

$$k = 2a + \frac{6b \times a}{b} = 8a \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8a}{m}}$$