Chapter Three

Relativistic Four Vectors

3.2 Minkowski Diagrams

Minkowski Diagrams (sometimes called "spacetime" diagrams) are extremely useful in seeing how coordinates transform between different reference frames.

Let frame S' move at speed v with respect to frame S (along the x - axis, as usual, and ignore the y and z components). Draw the x and ct axes of frame S. What do the x' and ct' axes of S' look like, superimposed on this diagram?

That is, at what angles are the axes inclined, and what is the size of one unit on these axes? (There is no reason why one unit on the x' and ct' axes should have the same length on the paper as one unit on the x and ct axes). We can answer these questions by using the Lorentz transformations, We'll first look at the ct' axis, and then the x' axis.

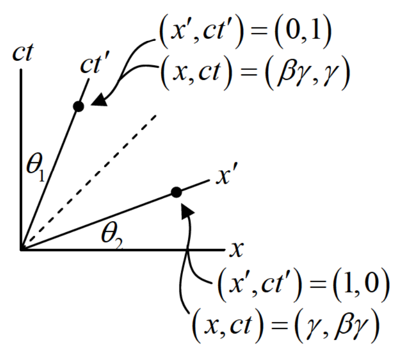
ct' - axis angle and unit size

Look at the point $(x', ct') = (0,1)$,

where $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ and $\beta = \dfrac{v}{c}$

Equation $x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{0+\frac{v}{c}.1}{\sqrt{1-\frac{v^2}{c^2}}}=\beta\gamma$

and $t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{\frac{1}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow ct=\gamma$



$(x',ct')=(0,1)\Rightarrow(x,ct)=(\gamma\beta,\gamma)$.

The angle between the ct' and ct axes is therefore given by $\tan\theta_1=\frac{\gamma\beta}{\gamma}=\beta$. With $\beta=\frac{v}{c}$, we have $\tan\theta_1=\beta$

Look at the point $(x',ct')=(1,0)$, Equation $x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1+0}{\sqrt{1-\frac{v^2}{c^2}}}=\gamma$ and

$$t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{0+\frac{v.1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\Rightarrow ct=\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}=\beta\gamma$$

$(x',ct')=(1,0)\Rightarrow(x,ct)=(\gamma,\gamma\beta)$

If θ_2 is angle between x and x' then $\tan\theta_2=\frac{\beta\gamma}{\gamma}=\beta$

Alternatively,

The ct' axis is simply the "worldline" of the origin of S' (A worldline is the path an object takes as it travels through spacetime). The origin moves at speed v with respect to S. Therefore, points on the ct' axis satisfy $x/t=v$, or $x/ct=v/c$.

On the paper, the point $(x',ct')=(0,1)$, which we just found to be the point $(x,ct)=(\gamma\beta,\gamma)$, is a distance $\gamma\sqrt{1+v^2/c^2}$ from the origin.

Therefore, using the definitions of β and γ, we see that $\frac{\text{one } ct' \text{ unit}}{\text{one } ct \text{ unit}}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ as measured on a grid where the x axes are orthogonal. This ratio approaches infinity as $\beta\to 1$. And it of course equals 1 if $\beta=0$.

x' - axis angle and unit size

The same basic argument holds here. Look at the point $(x',ct')=(1,0)$, which lies on the x' - axis, one x' unit from the origin. Equations $x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}=\gamma, t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{\gamma v}{c^2}\Rightarrow ct=\frac{\gamma v}{c}$ tell us that this point is the point $(x,ct)=(\gamma,\gamma\beta)$. The angle between the x' and x axes is therefore given by $\tan\theta_2=ct/x=v/c$. So as in the ct' - axis case, $\tan\theta_2=\beta$

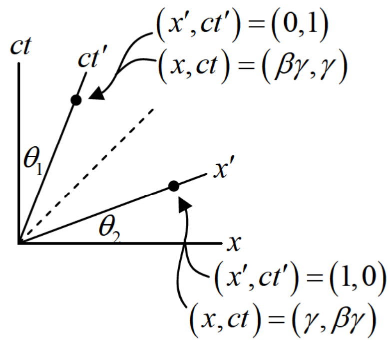


Figure 3.4

On the paper, the point $(x',ct')=(1,0)$, which we just found to be the point $(x,ct)=(\gamma,\gamma v/c)$, is a distance $\gamma\sqrt{1+v^2/c^2}$ from the origin. So, as in the ct' - axis case,

$$\frac{\text{one } x' \text{ unit}}{\text{one } x \text{ unit}}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

as measured on a grid where the x and ct axes are orthogonal. Both the x' and ct' axes are therefore stretched by the same factor, and tilted in by the same angle, relative to the x and ct axes. This "squeezing in" of the axes in a Lorentz transformation is different from what happens in a rotation, where both axes rotate in the same direction.