Chapter Three

Relativistic Four Vectors

3.3 Symmetry in Momentum and Energy

We have seen that $c^2t^2 - r^2 = c^2\tau_0^2$ is invariant under Lorentz transformation similarly

We have relation $E^2 = p^2c^2 + m_0^2c^4 \Rightarrow E^2 - p^2c^2 = m_0^2c^4$ which means rest mass m_0 is also invariant under Lorentz transformation

On the basis of symmetry, we can also discuss write Lorentz transformation for momentum vector $\vec{p}$ and energy E.

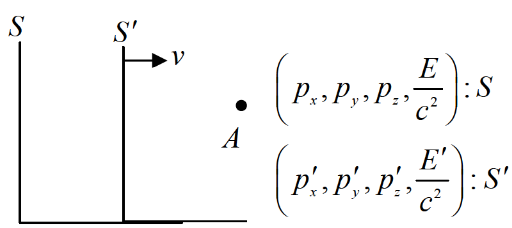


Figure 3.5

Ifmomentum p_x, p_y, p_z and energy E aremeasured from S frame. p'_x, p'_y, p'_z and E' are momentum and energy measured from S' frame

Thenthere are equivalent Lorentz transformation for energy and momentum as $\left(p_x, p_y, p_z, \frac{E}{c^2}\right)$ and $\left(p'_x, p'_y, p'_z, \frac{E'}{c^2}\right)$ which is given as

$$p_x = \frac{p'_x + \frac{vE'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \; p_y = p'_y \; p_z = p'_z \; \frac{E}{c^2} = \frac{\frac{E'}{c^2} + \frac{vp'_x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow E = \frac{E' + p'_x}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p'_x = \frac{p_x - \frac{vE}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \; p'_y = p_y \; p'_z = p_z \; \frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{vp_x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow E' = \frac{E - p_x}{\sqrt{1-\frac{v^2}{c^2}}}$$

3.4 Doppler Effect in Light

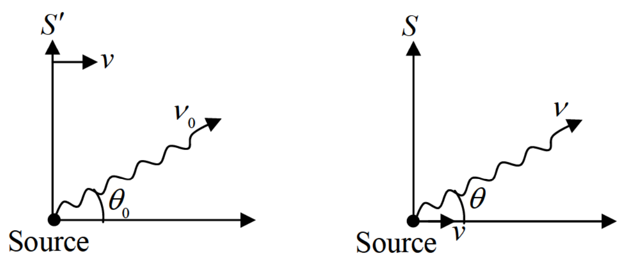


Figure 3.6

ν_0 is frequency measured by observer when there is not any relative speed between source and observer

v is the relative speed between source of light and observer

θ is the angle between velocity of light ($\vec{c} = c\cos\theta\hat{i} + c\sin\theta\hat{j}$) and relative velocity $\vec{v} = v\hat{i}$ between source and observer

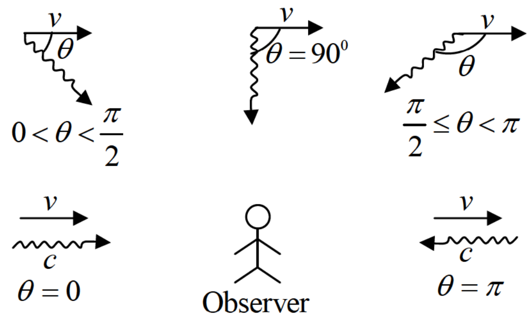
From S' frame i.e., when observer is attached to rest frame of source then $E = h\nu_0$

From S frame i.e., when observer and source has relative velocity

$E = h\nu$, $p_x = \frac{E}{c}\cos\theta = \frac{h\nu}{c}\cos\theta$,

Using Lorentz transformation between energy and momentum

$$E' = \frac{E - p_x}{\sqrt{1-\frac{v^2}{c^2}}} = h\nu_0 = \frac{h\nu - \frac{h\nu\cos\theta}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \nu = \frac{\nu_0\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta}$$



3.5 Longitudinal Doppler's effect

Case 1: $\theta = 0$ $\nu = \nu_0 \dfrac{\sqrt{1-\dfrac{v^2}{c^2}}}{1-\dfrac{v}{c}} = \nu_0 \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}$ for wave length $\lambda = \lambda_0 \sqrt{\dfrac{1-\dfrac{v}{c}}{1+\dfrac{v}{c}}}$

Case 2: $\theta = \pi$ $\nu = \nu_0 \dfrac{\sqrt{1-\dfrac{v^2}{c^2}}}{1+\dfrac{v}{c}} = \nu_0 \sqrt{\dfrac{1-\dfrac{v}{c}}{1+\dfrac{v}{c}}}$ for $\lambda = \lambda_0 \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}$

Transverse Doppler's effect

Case 3: $\theta = \dfrac{\pi}{2}$ $\nu = \nu_0 \sqrt{1-\dfrac{v^2}{c^2}}$ for $\lambda = \dfrac{\lambda_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

Example: Consider a radioactive nucleus that is travelling at a speed $\frac{c}{2}$ with respect to the lab frame. It emits γ - rays of frequency v_0 in its rest frame. There are two stationary detectors, (first detector A is not on the path of the nucleus in the lab .the second detector B is in path of nucleus such that nucleus approach it).

(a) If a γ - ray photon is emitted when the nucleus is closest to the detector, its observed frequency at the first detector is v_1, find the value of v_1

(b) The second detector observed frequency is v_2 find v_2

(c) What will measurement of frequency by detector B if nucleus is moving away from detector B?

Solution: (a) If detector A is not in path then closet distance is when nuclear is perpendicular to detector A so it is case of transverse Doppler's effect, so

$$v_1 = v_0\sqrt{1-\frac{v^2}{c^2}} \quad v_1 = v_0\sqrt{1-\frac{1}{4}} \Rightarrow v_1 = v_0\frac{\sqrt{3}}{2}$$

(b) Second detector B is in path of nucleus so there is longitudinal Doppler effect .The nuclear is moving towards the detector B so $v_2 = v_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = v_0\sqrt{\frac{1+\frac{1}{2}}{1-\frac{1}{2}}} = \sqrt{3}v_0$

(c) Second detector B is in path of nucleus so there is longitudinal Doppler effect .The nuclear is moving away the detector B so $v_2 = v_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = v_0\sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}} = \frac{v_0}{\sqrt{3}}$

Example: A distant galaxy in constellation Hydra is receding from the earth at $6.12\times10^7\ m/s$ by how much is a green spectral line of wavelength $500\,nm$ emitted by this galaxy shifted towards the red spectrum.

Solution: $\lambda = \lambda_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$ $v = 0.204c$ $\lambda_0 = 500\,nm$

$\lambda = 500\sqrt{\frac{1+.204}{1-.204}} = 615nm$ which is orange part of spectrum. The shift is $\lambda-\lambda_0 = 115\ nm$.

Properties of Four Vectors

(i) Linear Combinations: If A **and** B are 4-vectors, then $C = aA + bB$ is also a 4-vector

(ii) Inner Product invariance: Consider two arbitrary 4-vectors A **and** B. Define their inner product to be $A \cdot B = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 = A_0B_0 - \vec{A}\cdot\vec{B} = A'\cdot B'$, where A and B are measured from S frame and A' and B' are measured from S' frame.

(iii) Norms: For 4-vector A the norms is defined as

$$A^2 = A \cdot A = A_0A_0 - A_1A_1 - A_2A_2 - A_3A_3 = A_0^2 - |A|^2$$

Four Vectors and Relativistic Invariance

- Four position vectors $ds = (dx, dy, dz, icdt)$
- Four velocity vectors $\frac{ds}{dt} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{icdt}{d\tau}\right)$ $u = \gamma\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{icdt}{dt}\right)$

  $u = \gamma(u_x, u_y, u_z, ic) \Rightarrow u = \gamma(\vec{u}, ic)$
- Four momentum, Four Energy vector: $P = \gamma(m_0u, im_0c)$

  $$= (m\vec{u}, imc) = \left(\vec{p}, \frac{imc^2}{c}\right) \Rightarrow \vec{P} = \left(\vec{p}, \frac{iE}{c}\right)$$
- Four Force: $F = \gamma\left(\vec{F}, \frac{idmc}{dt}\right)$
- Four dimensional space time continuum:

  The square of interval is represented as

  $S_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = r_{12}^2 - c^2(t_2 - t_1)^2$

Collisions and Decays

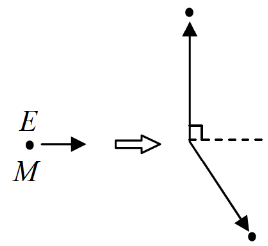
One can use concept of 4-vector as conservation of energy and momentum in decay or collision problem.

4-vector momentum $\vec{P}$ can be defined as $\vec{P} = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right)$,

where $\left(\frac{E}{c}\right)^2 - |p|^2 = m^2c^2 \Rightarrow \vec{P}\cdot\vec{P} = m^2c^2$.

Conservation and energy and momentum in a collision or decay problem reduce to concise statement $\vec{P}_{before} = \vec{P}_{after}$

Example: A particle with mass M and energy E decays into two identical particles. In the lab frame, one of them is emitted at a 90^0 angle, as shown in figure. What are the energies of the created particles? (Take $c=1$)



Solution: The 4-momentum before the decay is

$$P=(E,p,0,0)$$

where $p=\sqrt{E^2-M^2}$. Let the created particles have mass m, and let the second particle make an angle θ with the x- axis. The 4-momenta after the decay are

$$P_1=(E_1,0,p_1,0),\ P_2=(E_2,p_2\cos\theta,-p_2\sin\theta,0)$$

With using the 4-momenta, conservation of energy and momentum can be combined into the statement, $\vec{P}=\vec{P}_1+\vec{P}_2$. Therefore,

$$P-P_1=P_2\Rightarrow(P-P_1)\cdot(P-P_1)=P_2\cdot P_2\Rightarrow P^2-2P\cdot P_1+P_1^2=P_2^2$$

$$\Rightarrow M^2-2EE_1+m^2=m^2\Rightarrow E_1=\frac{M^2}{2E}$$

And then $E_2=E-E_1=(2E^2-M^2)/2E$. This solution should convince you that 4-momenta can save you a lot of work. What happened here was that the expression for P_2 was fairly messy, but we arranged things so that it appeared only in the form of P_2^2, which is simply m^2. 4-momenta provide a remarkably organized method for sweeping unwanted garbage under the rug.