Chapter Four

Relativistic Electrodynamics

4.2 Transformation of electric and magnetic field

In one inertial frame S the charge density and current density of system measured. If the same system is measured from another inertial frame S' the charge and current density of same system is different. This transformation is according to Lorentz transformation. Hence charge density and current density are transformed it is very obvious that electric field and magnetic field will also be transformed.

If component of electric field $\vec{E}':\left(E'_x, E'_y, E'_z\right)$ and component of magnetic field $\vec{B}':\left(B'_x, B'_y, B'_z\right)$ are measured from S' frame then these component of electric field $\vec{E}:\left(E_x, E_y, E_z\right)$ and component of magnetic field $\vec{B}:\left(B_x, B_y, B_z\right)$ are measured from S.

The electric field is transformed as $E'_{\parallel} = E_{\parallel}$ $E'_{\perp} = \gamma\left[E_{\perp} + \left(v \times B\right)_{\perp}\right]$

$$E'_x = E_x \qquad E_x = E'_x$$

$$E'_y = \gamma\left(E_y - vB_z\right) \qquad E_y = \gamma\left(E'_y + vB'_z\right)$$

$$E'_z = \gamma\left(E_z - vB_y\right) \qquad E_z = \gamma\left(E'_z + vB'_y\right)$$

$B'_{\parallel} = B_{\parallel}$ and $B'_{\perp} = \gamma\left[B_{\perp} - \frac{1}{c^2}(v \times E)_{\perp}\right]$

$B'_x = B_x \qquad B_x = B'_x$

$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad B_y = \gamma\left(B'_y - \frac{v}{c^2}E'_z\right)$

$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right), \quad B_z = \gamma\left(B'_z - \frac{v}{c^2}E'_y\right)$

Example: A test charge is rest in frame S outside a long straight stationary wire carrying a current. . The density of free electrons is n and drift velocity of electrons in wire is u

(a) Find the net current and net charge in the wire from S frame

(b) Find the electric filed and magnetic field measured by observer from the rest frame of S at distance r perpendicular distance

(c) if another frame S' moving with respect to S along length of wire, find electric field and magnetic field measured from rest frame of S'

Solution: (a) With respect to rest frame of S positive ions are rest but linear charge density of electron λ^- and linear charge density of positive ions λ^+ are same so there is no any net charge density. So, total charge density $\lambda = 0$. The current in rest frame of S frame is $i = \lambda^- u = -neu$

(b) With respect to rest frame of wire or from S frame, the total charge density is $\lambda = 0$, so electric field is zero i.e., $E_x = 0, E_y = 0, E_z = 0$

and magnetic field is $B_x = 0,\ B_y = 0\ B_z = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 neu}{2\pi r}$

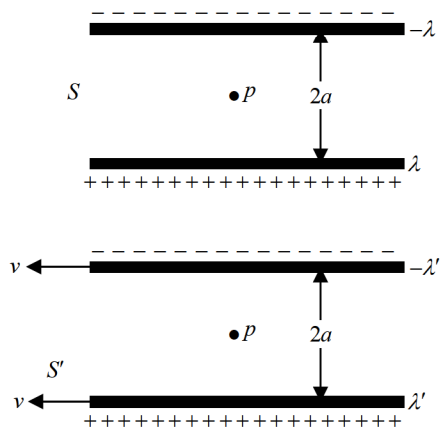
(c) $E'_x = E_x\ E'_y = \gamma\left(E_y - vB_z\right), E'_z = \gamma\left(E_z - vB_y\right)$

so $\quad E'_x = 0,\ E'_y = -v\gamma B_z = -\gamma\frac{\mu_0 nevu}{2\pi r}\ E'_z = 0$

$B'_x = B_x\ B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)\ B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$

$B'_x = B_x = 0\ B'_y = 0\ B'_z = \gamma\frac{\mu_0 neu}{2\pi r}$

Example: consider two long parallel wires separated by a distance $2a$ bearing equal and opposite uniform charge distributions. In frame S at rest with respect to wires there is no current flowing and the linear charge density is λ



(a) Calculate the electric and magnetic fields in frame S at point P mid away between wires

(b) Frame S' moves at velocity parallel to length of wires ($x-axis$). Find the linear charge density λ' and current in the wires as measured in frame S'.

(c) Calculate the electric and magnetic fields measured in frame S' at mid away between wires.

(d) Verify the result from transformation of electric filed and magnetic field.

Solution: (a) $E_x = 0, E_y = \frac{-\lambda}{2\pi\varepsilon_0 a}(-\hat{y}) + \frac{\lambda}{2\pi\varepsilon_0 a}\hat{y} = \frac{\lambda}{\pi\varepsilon_0 a}\hat{y}\ E_z = 0$

(b) $\lambda' = \frac{\lambda - \frac{vj_x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\lambda}{\sqrt{1-\frac{v^2}{c^2}}}$ because $j_x = 0$

$$j_x' = \frac{j_x - \lambda v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{-\lambda v}{\sqrt{1-\frac{v^2}{c^2}}}$$

(c) $E_x' = 0\ E_y' = \frac{-\lambda'}{2\pi\varepsilon_0 a}(-\hat{y}) + \frac{\lambda'}{2\pi\varepsilon_0 a}\hat{y} = \frac{\lambda'}{\pi\varepsilon_0 a}\hat{y} \Rightarrow E_y' = \frac{\lambda}{\pi\varepsilon_0 a\sqrt{1-\frac{v^2}{c^2}}}\hat{y}$

$$B_x' = 0, B_y' = 0\ B_z' = \frac{\mu_0 i'}{2\pi a}\hat{z} + \frac{\mu_0 - i'}{2\pi a} - \hat{z} = -\frac{\mu_0 \lambda v}{\pi a\sqrt{1-\frac{v^2}{c^2}}}\hat{z}$$

(d) $E_x' = E_x = 0$, $E_z' = \gamma\left(E_z - vB_y\right) = 0$

$$E'_y = \gamma\left(E_y - vB_z\right) = \gamma\frac{\lambda}{\pi\varepsilon_0 a} \Rightarrow E'_y \frac{\lambda}{\pi\varepsilon_0 a\sqrt{1-\frac{v^2}{c^2}}},$$

$$B'_x = B_x = 0,\ B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \Rightarrow B'_y = 0$$

$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right), \Rightarrow B'_z = -\gamma\frac{v}{c^2}\frac{\lambda}{\pi\varepsilon_0 a} \Rightarrow B'_z = -\frac{\mu_0\lambda v}{\pi a\sqrt{1-\frac{v^2}{c^2}}} \text{ put } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

List of Lorentz Invariant Quantity:

- Proper time τ_0 is Lorentz invariant $c^2\tau^2 - |\vec{r}|^2 = c^2\tau_0^2$ where τ is time and $\vec{r}$ position vector
- Rest mass m_0 is Lorentz invariant $c^2m^2 - |\vec{p}|^2 = c^2m_0^2$ where m is relativistic mass and $\vec{p}$ is associated relativistic momentum vector
- Rest mass energy m_0c^2 is Lorentz invariant $E^2 - |\vec{p}|^2 c^2 = m_0^2c^4$ where E is total relativistic energy and $\vec{p}$ is associated momentum vector
- Rest charge ρ_0 is Lorentz invariant $c^2\rho^2 - |\vec{j}|^2 = c^2\rho_0^2$ where ρ is charge density and $\vec{j}$ is associated current density
- $|\vec{E}|^2 - c^2|\vec{B}|$ and $\vec{E}.\vec{B}$ are Lorentz invariant where $\vec{E}$ is electric field and $\vec{B}$ is magnetic field
- D'Alembert operator $\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is Lorentz invariant where ∇^2 is Laplacian operator

Example: A inertial observer A at rest measures the electric and magnetic field $E = (\alpha, 0, 0)$ and $B = (\alpha, 0, 2\alpha)$ in a region, where a is a constant. Another inertial observer B, moving with a constant velocity with respect to A, measures the fields as $E' = (E'_x, \alpha, 0)$ and $B' = (\alpha, B'_y, \alpha)$. Then in units $c = 1$, find E'_x and B'_y are given, respectively.

Solution: for given value of electric field $\vec{E}$ and $\vec{B}$ the quantity $\vec{E}.\vec{B}\ |\vec{E}|^2 - |\vec{B}|^2$ is invariant

For inertial observer A electric field is $\vec{E} = \alpha\hat{i}$ and magnetic field is $\vec{B} = \alpha\hat{i} + 2\alpha\hat{k}$

$\vec{E}.\vec{B} = \alpha^2$ and $|\vec{E}|^2 - |\vec{B}|^2 = \alpha^2 - \alpha^2 + 4\alpha^2 = |\vec{E}|^2 - |\vec{B}|^2 = 4\alpha^2$

Now for inertial observer B,

electric field is $\vec{E}' = E'_x\hat{i} + \alpha\hat{j}$ and magnetic field is $\vec{B}' = \alpha i + B'^2_y\hat{j} + \alpha\hat{k}$

So $\vec{E}' \cdot \vec{B}' = E'_x \alpha + B'_y \alpha$ and $\left|E'^2\right| - \left|B'^2\right| = \left(E'^2_x + \alpha^2\right) - \left(\alpha^2 + B'^2_y + \alpha^2\right) = E'^2_x - B'^2_y - \alpha^2$

$\vec{E}.\vec{B} = \vec{E}'.\vec{B}' \Rightarrow \alpha^2 = E'_x \alpha + B'_y \alpha \Rightarrow E'_x + B'_y = \alpha$

$\left|\vec{E}\right|^2 - \left|\vec{B}\right|^2 = \left|E'^2\right| - \left|B'^2\right| \Rightarrow 4\alpha^2 = E'^2_x - B'^2_y - \alpha^2 \Rightarrow E'^2_x - B'^2_y = -3\alpha^2$

Solving $E'^2_x - B'^2_y = -3\alpha^2$ and $E'_x + B'_y = \alpha$ we get $E'_x = -\alpha$, and $B'_y = 2\alpha$