Solution (Class Test)

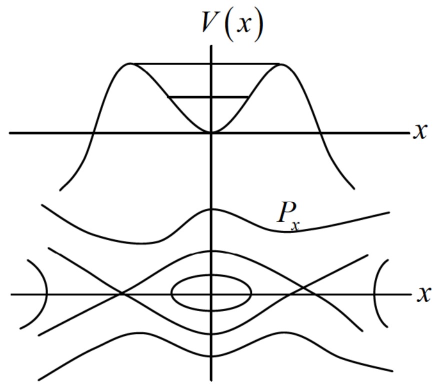
Topic: Mechanics (Stability Analysis)

Ans. 1: (a)

Solution: $V(x) = -k^2 x^4 + \omega^2 x^2$

For equation point

$\frac{\partial V}{\partial x} = 0 \Rightarrow -4k^2x^3 + 2\omega^2 x = 0$, $x = 0$ or $x^2 = \frac{\omega^2}{2k^2}$



Now, $\frac{d^2V}{dx^2} = -12k^2x^2 + 2\omega^2$ At, $x = 0$

$\frac{d^2V}{dx^2} = 2\omega^2, x = 0$ is minimum.

At, $x^2 = \frac{\omega^2}{2k^2}$

And, $\frac{d^2V}{dx^2} = -12k^2 \frac{\omega^2}{2k^2} + 2\omega^2 = -4\omega^2$

Hence, $x = \pm\sqrt{\frac{\omega^2}{2k^2}}$ is maxima.Both bounded and unbounded motions are possible

Ans. 2: (c)

Solution: At $x = x_0$ potential energy is minimum. Since total energy is constant therefore, K. E. is maximum. So momentum is maximum.

From Taylor's expansion $V(x)$ can be approximated as

$$V(x) = V(x_0) + (x - x_0)\frac{\partial V}{\partial x}\bigg| + \frac{(x-x_0)^2}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x=0} + ...$$

$(x - x_0)^3 = (x - x_0)^4 = ...$ is very small.

So it can be zero.

$$\therefore V(x) = V(x_0) + (x - x_0)\frac{\partial V}{\partial x}\bigg|_{x=x_0} + (x - x_0)^2 \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x=0}$$

hence $V(x)$ is min $\frac{\partial V}{\partial x}\bigg| = 0$

so force $= 0$

hence $V(x) = V(x_0) + (x - x_0)^2 \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x=0}$ so $k = \frac{\partial^2 V}{\partial x^2}\bigg|_{x=x_0}$

Ans. 3: (b)

Solution: $V(x)=x(x-4)^2$ for equilibrium point

$$\frac{\partial V}{\partial x}=0\Rightarrow (x-4)^2+2x(x-4)=0\Rightarrow (x-4)(3x-4)=0$$

$$x=4,\ x=\frac{4}{3}$$

$$\frac{\partial^2 V}{\partial x^2}=(3x-4)+3(x-4)=6x-16$$

At $x=4$, $\frac{\partial^2 V}{\partial x^2}=6x-16=8>0$; which is stable equilibrium point

and at $x=\frac{4}{3}$, $\frac{\partial^2 V}{\partial x^2}=6x-16=-4<0$, so it is an unstable equilibrium point.

$$\omega=\sqrt{\frac{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=4}}{m}}=\sqrt{\frac{8}{1}}=\sqrt{8}$$

Ans. 4: (a)

Solution: $V(x)=ax+\frac{b}{x}\Rightarrow \frac{\partial V}{\partial x}=0\Rightarrow a-\frac{b}{x^2}=0\ ax^2-b=0\Rightarrow x_0=\pm\left(\frac{b}{a}\right)^{\frac{1}{2}}$.

Since $\omega=\sqrt{\frac{k}{m}}$, $\mathrm{m}=1$ and $k=\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}$ where x_0 is stable equilibrium point.

Hence $k=\frac{\partial^2 V}{\partial x^2}=\frac{2b}{x_0^3}=2\sqrt{\frac{a^3}{b}}$.

Thus $\omega=\sqrt{2}\left(\frac{a^3}{b}\right)^{\frac{1}{4}}$.

Ans. 5: (a)

Solution: $\frac{dU}{dx}=0\Rightarrow k\left(6x^2-10x+4\right)=0\Rightarrow x=1,\ x=\frac{2}{3}$

$$\frac{d^2U}{dx^2}=k(12x-10)$$

For $x=1$, $\frac{d^2U}{dx^2}=2k>0$ and for $x=\frac{2}{3}$, $\frac{d^2U}{dx^2}=-2k<0$

So $\omega=\sqrt{\frac{2k}{m}}\Rightarrow \frac{2\pi}{T}=\sqrt{\frac{2k}{m}}\Rightarrow T=2\pi\sqrt{\frac{m}{2k}}\Rightarrow \pi\sqrt{\frac{2m}{k}}$

Ans. 6: (a)

Solution: $\because V(x) = -\frac{1}{4}ax^4 + \frac{1}{6}bx^6$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow -ax^3 + bx^5 = 0 \Rightarrow x^3\left[-a + bx^2\right] = 0 \Rightarrow x = \pm\left(\frac{a}{b}\right)^{\frac{1}{2}}, 0$$

$$\because \frac{\partial^2 V}{\partial x^2} = -3ax^2 + 5bx^4 \Rightarrow At \quad x = 0, \frac{\partial^2 V}{\partial x^2} = 0$$

Thus $x = 0$ is a saddle point.

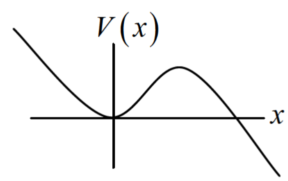
$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}} = -3a.\frac{a}{b} + 5b\frac{a^2}{b^2} = \frac{2a^2}{b}$$ (Positive, so it is a stable point)

$$\Rightarrow \omega = \sqrt{\frac{\frac{\partial^2 V}{\partial x^2}}{m}} = \sqrt{\frac{2a^2}{bm}}$$

Ans. 7: (b)

Solution: $\frac{dV}{dx} = 0 \Rightarrow x - x^2 = 0 \Rightarrow x = 0,\ x = 1$ are equilibrium points.

$$\frac{d^2V}{dx^2} = 1 - 2x$$



At $x = 0,\ \frac{d^2V}{dx^2} = 1 > 0$, which is stable equilibrium point.

So, it is minima of the curve.

At $x = 1,\ \frac{d^2V}{dx^2} = -1 < 0$, which is unstable equilibrium point. So it is maxima of the curve.

Ans. 8: (c)

Ans. 9: (b)

Solution: $V = \frac{1}{2}kx^2 - \frac{1}{4}\lambda x^4$

$$\frac{dV}{dx} = 0 \Rightarrow kx - \lambda x^3 = 0$$

$$x = 0,\quad x^2 = \frac{k}{\lambda} \Rightarrow x = x_0 = \sqrt{\frac{k}{\lambda}}$$

$\frac{d^2V}{dx^2} = k$ at $x = 0$ so $x = 0$ is stable equilibrium point $\frac{d^2V}{dx^2} = -2k$ at $x_0 = \sqrt{\frac{k}{\lambda}}$

so $x_0 = \sqrt{\frac{k}{x}}$ is unstable equilibrium point $\omega = \sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x=0}}{m}} = \sqrt{\frac{k}{m}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Ans. 10: (d)

Solution: $E = \frac{p_x^2}{2m} - kx^2$ which is shape of hyperbola

Ans. 11: (a)

Ans. 12: (d)

Solution: $V(x) = \left(\sqrt{x^2 + l^2} - l\right)^2$

Using Taylor expansion $V(x) = \left(\sqrt{x^2 + l^2} - l\right)^2 = l^2\left(\sqrt{1 + \left(\frac{x}{l}\right)^2} - 1\right)^2 = l^2\left(1 + \frac{1}{2}\left(\frac{x}{l}\right)^2 \ldots - 1\right)^2$

so $V(x) \propto x^4$

Ans. 13: (d)

Solution: Amplitude is $A = \sqrt{\frac{2E}{k}}$

Ans. 14: (a)

Ans. 15: (c)