Chapter One
FOURIER SERIES

1.4 Complex Fourier Series

The complex Fourier series will be presented first with period 2π, then with general period $2L$. The connection with the real-valued Fourier series will be explained then and formulae will be given for converting between the two types of representation.

Then examples are given of computing the complex Fourier series and converting between complex and real series.

New Basis Functions

Recall that the Fourier series builds a representation composed of a weighted sum of the following basis functions.

1. constant term
2. $\cos t \;\; \cos 2t \;\; \cos 3t \;\; \cos 4t \ldots$
3. $\sin t \;\; \sin 2t \;\; \sin 3t \;\; \sin 4t \ldots$

Computing the weights a_n, b_n and a_0 often involves some nasty integration.

We now present an alternative representation based on a different set of basis functions:

1 (i.e., a constant term)

e^{it} e^{2it} e^{3it} e^{4it} ...

e^{-it} e^{-2it} e^{-3it} e^{-4it} ...

These can all be represented by the term e^{int} with n taking integer values from $-\infty$ to $+\infty$.

Note that the constant term is provided by the case when $n = 0$.

Series of Complex Exponentials

A representation based on this family of functions is called the "complex Fourier series".

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

The coefficients, c_n, are normally complex numbers.

It is often easier to calculate than the $\sin/\cos$ Fourier series because integrals with exponentials in are usually easy to evaluate. The beauty and power of Euler's Equation.

We will now derive the complex Fourier series equations, as shown above, from the sin/cos Fourier series using the expressions for $sin()$ and $\cos()$ in terms of complex exponentials.

Complex Fourier Series

$$f(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos nt + b_n \sin nt] = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{int}+e^{int}}{2}\right) + b_n\left(\frac{e^{int}-e^{-int}}{2i}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\frac{(a_n - ib_n)}{2}e^{int} + \sum_{n=1}^{\infty}\frac{(a_n + ib_n)}{2}e^{-int} = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Where, $c_n = \begin{cases} a_0 & , n = 0 \\ (a_n - ib_{-n})/2 & , n = 1, 2, 3, \ldots \\ (a_{-n} + ib_{-n})/2 & , n = -1, -2, -3, \ldots \end{cases}$

Note: a_{-n} and b_{-n} are only defined when n is negative.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\, f(t)\, dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)\, f(t)\, dt$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

Thus, for n positive

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(nt) - i\sin(nt)]\, f(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\, f(t)\, dt$$

for n negative

$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(-nt) + i\sin(-nt)]f(t)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt$$

and for $n = 0$

$$c_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-0} f(t)dt$$

Complex Fourier Series Summary

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Example: Find the complex Fourier series to model $f(t) = \sin(t)$

Solution: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\sin(t)\,dt = \frac{1}{2\pi}\left[\frac{e^{in\pi} - e^{-in\pi}}{n^2-1}\right]$

which is zero when n does not equal 1 or −1. For these two special cases we set $n = 1 + \varepsilon$ and calculate the limit of c_n as ε tends to zero. This gives us

$$c_1 = \frac{1}{2i} \; and \; c_{-1} = \frac{-1}{2i}$$

Which means the complex Fourier series for $f(t) = \sin(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{e^{it} - e^{-it}}{2i}$$

Finding the limit as $n \to 1$

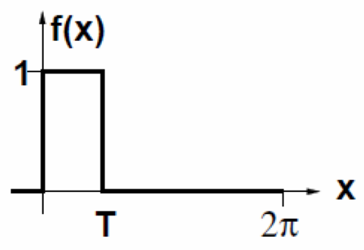
$$c_n = \frac{1}{2\pi}\left[\frac{e^{in\pi} - e^{-in\pi}}{n^2 - 1}\right]$$

Set $n = 1 + \varepsilon$ and let ε tend to zero

$$c_1 = \frac{1}{2\pi}\left[\frac{e^{i\pi(1+\varepsilon)} - e^{-i\pi(1+\varepsilon)}}{(1+\varepsilon)^2 - 1}\right] = \frac{1}{2\pi}\left[\frac{-e^{i\pi\varepsilon} + e^{-i\pi\varepsilon}}{(1+\varepsilon)^2 - 1}\right] \approx \frac{1}{2\pi}\left[\frac{-1 - i\pi\varepsilon + 1 - i\pi\varepsilon}{1 + 2\varepsilon - 1}\right]$$

$$\approx \frac{1}{2\pi}\left[\frac{-2i\pi\varepsilon}{2\varepsilon}\right] \approx \frac{-i}{2} \approx \frac{1}{2i}$$

Example: Find the complex Fourier series to model $f(x)$ that has a period of 2π and is 1 when $0 < x < T$ and zero when $T < x < 2\pi$.



Solution: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(t)\,dt = \frac{i}{2\pi n}\left[e^{-inT} - 1\right], when\ n \neq 0 = \frac{1}{2\pi}\ area = \frac{T}{2\pi},\ when\ \ n = 0$

So, the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1}\frac{1}{n}\left[e^{-inT} - 1\right]e^{int} + \sum_{n=1}^{\infty}\frac{i}{n}\left[e^{-inT} - 1\right]e^{int}\right\}$$

Converting c to a, b, and d

From our example,

$$c_n = \begin{cases} \frac{i}{2\pi n}\left[e^{-inT} - 1\right] & , when\ n \neq 0 \\ \frac{1}{2\pi} area = \frac{T}{2\pi} & , when\ n = 0 \end{cases}$$

We wish to calculate the coefficients for the equivalent. Fourier series in terms of $sin()$ and $cos()$.

Clearly, $a_0 = c_0 = \frac{T}{2\pi}$. For $n > 0$

$$c_n = (a_n - ib_n)/2 \Rightarrow a_n = 2\,\mathrm{Re}\,\{c_n\},\ b_n = -2\,\mathrm{Im}\,\{c_n\}$$

converting our expression for c_n into $sin()$ and $cos()$:

$$2c_n = \frac{i}{\pi n}[\cos(nT) - i\sin(nT) - 1] = \frac{1}{\pi n}[\sin(nT) + i(\cos(nT) - 1)]$$

So, $a_n = \frac{\sin(nT)}{n\pi}$ *and* $b_n = \frac{1 - \cos(nT)}{n\pi}$.

Complex Fourier Series

$$f(t) = \frac{1}{2\pi}\left\{T + \sum_{n=-\infty}^{-1}\frac{i}{n}\left[e^{-inT} - 1\right]e^{int} + \sum_{n=1}^{\infty}\frac{i}{n}\left[e^{-inT} - 1\right]e^{int}\right\}$$

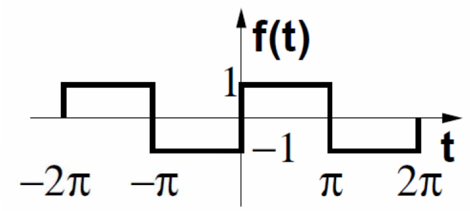
Real Fourier Series

$$f(t)=\frac{T}{2\pi}+\sum_{n=1}^{\infty}\frac{\sin\,(nT)}{n\pi}\cos\,(nt)+\sum_{n=1}^{\infty}\frac{1-\cos(nT)}{n\pi}\sin\,(nt)$$

Both series converge as $1/n$.

Converting from Real to Complex

Convert the real Fourier series of the square wave $f(t)$ to a complex series.



For the real series, we know that $a_0=a_n=0$ and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nt)\;f(t)\;dt=\frac{4}{n\pi},\;n\;odd$$

Giving $f(t)=\frac{4}{\pi}\left[\sin(t)+\frac{\sin(3t)}{3}+\frac{\sin(5t)}{5}+\ldots\right]$

To convert to a complex series, use

$$c_n=\begin{cases}a_0 & ,\;n=0\\ (a_n-ib_n)/2 & ,\;n=1,2,3,\ldots\\ (a_{-n}+ib_{-n})/2 & ,\;n=-1,-2,-3,\ldots\end{cases}$$

so, we have

$$c_0=0$$

$$c_n=-2i/(n\pi)\;,\;n\text{ positive and odd}$$

$$c_n=2i/(-n\pi)\;,\;n\text{ negative and }|n|\text{ odd}$$

$$\Rightarrow f(t)=\frac{-2i}{\pi}\left[\ldots+\frac{e^{-5it}}{-5}+\frac{e^{-3it}}{-3}+\frac{e^{-it}}{-1}+\frac{e^{it}}{1}+\frac{e^{3it}}{3}+\frac{e^{5it}}{5}+\ldots\right]$$

General Complex Series

For period of 2π

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}e^{-\text{int}}f(t)\;dt$$

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{\text{int}}$$

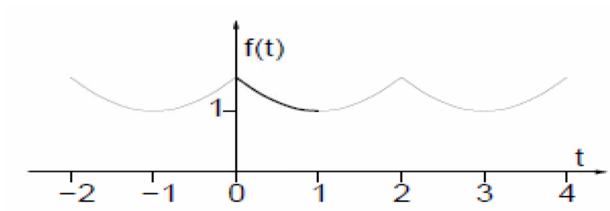
Similarly, for period L

$$c_n = \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}$$

The fraction $\frac{2\pi}{L}$ is often written as ω_0 and called the fundamental angular frequency.

Example: An even function $f(t)$ is periodic with period $L = 2$, and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$. Find a complex Fourier series representation for $f(t)$.



$$c_n = \frac{1}{L}\int_0^L e^{-\text{int}\frac{2\pi}{L}} f(t)\,dt = \frac{1}{2}\int_0^2 e^{-\text{int}\,\pi}\cosh(t-1)\,dt = \frac{\sinh(1)}{1+n^2\pi^2}$$

Hence the complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\text{int}\frac{2\pi}{L}} = \sum_{n=-\infty}^{\infty} \frac{\sinh(1)\,e^{\text{int}\,\pi}}{1+n^2\pi^2}$$

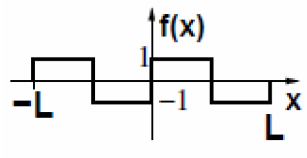
We can check this answer by computing the equivalent real Fourier series,

$$a_n = 2\,\text{Re}\{c_n\} \qquad ,n = 1,2,3,\ldots$$
$$b_n = -2\,\text{Im}\{c_n\} \qquad ,n = 1,2,3,\ldots$$

In this case, as c_n is entirely real,

$$a_n = 2c_n = \frac{2\sinh(1)}{1+n^2\pi^2}, n = 1,2,3,\ldots$$
$$b_n = 0$$
$$a_0 = \sinh(1)$$

Example: Find the complex Fourier series of the square wave $f(x)$



Note that the mean of the function is zero, so $c_0 = 0$.

$$c_n = \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\, dx = \frac{1}{L}\left[\int_0^{L/2} e^{-inx\frac{2\pi}{L}} dx - \int_{L/2}^{L} e^{-inx\frac{2\pi}{L}} dx\right]$$

$$= \frac{1}{2in\pi}\left[e^{-2in\pi} + 1 - 2e^{-in\pi}\right]$$

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\left[1-e^{-in\pi}\right]}{in\pi} 2\, e^{inx\frac{2\pi}{L}}$$

$$f(x) = \frac{2}{i\pi}\left[\ldots + \frac{e^{-5ix\frac{2\pi}{L}}}{-5} + \frac{e^{-3ix\frac{2\pi}{L}}}{-3} + \frac{e^{-ix\frac{2\pi}{L}}}{-1} + \frac{e^{ix\frac{2\pi}{L}}}{1} + \frac{e^{3ix\frac{2\pi}{L}}}{3} + \frac{e^{5ix\frac{2\pi}{L}}}{5} + \ldots\right]$$

Converting to a Real Series

We wish to convert the complex general range square wave series into a series with real coefficients.

$$c_n = \begin{cases} 2/(in\pi), & |n|\ odd \\ 0 & ,|n|\ even \end{cases}$$

Clearly $a_0 = c_0 = 0$. For a and b use:

$$\begin{aligned} c_n &= (a_n - ib_n)/2 \\ \Rightarrow a_n &= 2\,\mathrm{Re}\{c_n\} = 0 \\ and\ b_n &= -2\ \mathrm{Im}\ \{c_n\} = \frac{4}{n\pi},\ n\ odd \end{aligned}$$

Which gives us the real series:

$$f(t) = \frac{4}{\pi}\left[\sin\left(x\frac{2\pi}{L}\right) + \frac{\sin\left(3x\frac{2\pi}{L}\right)}{3} + \frac{\sin\left(5x\frac{2\pi}{L}\right)}{5} + \ldots\right]$$

Summary

For period L

$$c_n = \frac{1}{L}\int_0^L e^{-inx\frac{2\pi}{L}} f(x)\, dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{2\pi}{L}}$$

Relationship with the cos/sin Fourier series.

$$c_n = \begin{cases} d & ,n=0 \\ (a_n - ib_n)/2 & ,n=1,2,3,\ldots \\ (a_{-n}+ib_{-n})/2 & ,n=-1,-2,-3,\ldots \end{cases}$$

$$a_n = 2\,\mathrm{Re}\{c_n\} \qquad ,n=1,2,3,\ldots$$

$$b_n = -2\,\mathrm{Im}\{c_n\} \qquad ,n=1,2,3,\ldots$$

$$a_0 = c_0$$