Chapter One
FOURIER SERIES

1.1 Full Series

The function $f(x)$ can be written in the form of infinite trigonometric series,

$$\boxed{f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)}$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \qquad n=1,2,....$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx \qquad n=1,2,....$$

In another way for cosmetic reasons (see the symmetry $1/\pi$ in each of the coefficients) the FS can be written as.

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)}$$

Fourier coefficients are given by Euler formulas,

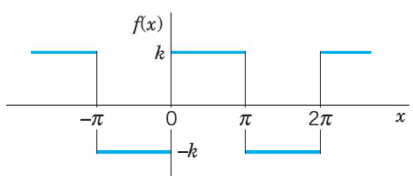
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad n = 1,2,....$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n = 1,2,....$$

Example: Write the Fourier series for $f(x)$.

$$f(x) = \begin{cases} -k & if \quad -\pi < x < 0 \\ k & if \quad 0 < x < \pi \end{cases} \quad and \quad f(x+2\pi) = f(x)$$



Given function $f(x)$ (Periodic reactangular wave)

Physically: Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

Note: Discontinuity:- The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.

Solution: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_o = 0$. (See the symmetry in graph behaves like odd function)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad \because f(x)\cos nx = \text{Odd Function}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} k\sin nx\,dx = \frac{2k}{n\pi}(1-\cos n\pi). \quad \because f(x)\cos nx = \text{Even Function}$$

Hence the Fourier coefficients b_n of our function are

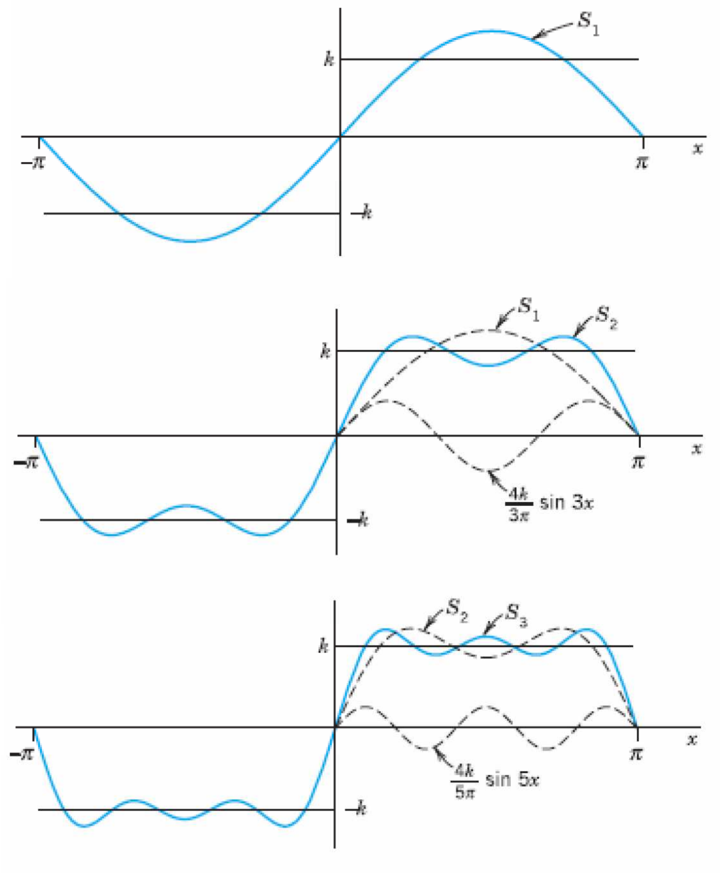
$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0,$$

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right).$$

Leibniz Series (1667): The famous Leibnitz series can be derived by setting $x = \pi/2$ we have,

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + ...\right).$$

Thus, $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - .... = \frac{\pi}{4}.$$



Orthogonality of the Trigonometric System

The trigonometric system is orthogonal on the interval (hence $-\pi \le x \le \pi$ also on $0 \le x \le 2\pi$ or any other interval of length 2π because of periodicity); that is, the integral of the product of any two functions in over that interval is 0, sothat for any integers n and m,

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \ne m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \ne m)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \ne m \text{ or } n = m)$$

Convergence and Sum of a Fourier Series

The class of functions that can be represented by Fourier series is surprisingly large and general. Sufficient conditions that are valid in most applications are as follows:

Representation by a Fourier Series

(a) Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$.

(b) Let $f(x)$ have a left-hand derivative and a right hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges.

(c) Its sum is $f(x)$, except at points x_0 where it is discontinuous.

Note: At the point of discontinuity**,** the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Arbitrary Period.

Transition from period 2π to any period $2L$, for the function $f(x)$, simply by a transformation of scale on the x - axis.

From period 2π to any period $p=2L$

Derivation**:** Periodic functions in applications may have any period not just 2π.

The transition from period 2π to be period $p=2L$ is affected by a suitable change of scale. Let $f(x)$ have period $p=2L$. Then we can introduce a new variable v such that, as a function of v, has period $f(x)$. If we set

$$x=\frac{p}{2\pi}v \text{, so that } v=\frac{2\pi}{p}x=\frac{\pi}{L}x$$

Then $v=\pm\pi$ corresponds to $x=\pm L$. This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$f(x)=f\left(\frac{L}{\pi}v\right)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nv+b_n\sin nv\right)$$

With coefficients,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(\frac{L}{\pi}v\right)dv, \qquad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{L}{\pi}v\right)\cos nv dv$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{L}{\pi}v\right)\sin nv dv$$

We could use these formulas directly, but the change to x simplifies calculations. Since $v=\frac{\pi}{L}x$, we have $dv=\frac{\pi}{L}dx$

And we integrate over x from $-L$ to L. Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right)}$$

With the Fourier coefficients of $f(x)$ given by the Euler formulas (π/L in dx cancels $1/\pi$)

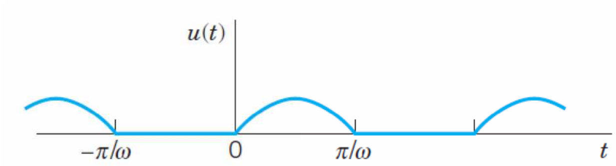
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, \ldots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, \ldots.$$

Example: Half-wave rectifier

A sinusoidal voltage $E\sin\omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Write its FS.



$$u(t) = \begin{cases} 0 & if \quad -L < t < 0, \\ E\sin\omega t & if \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \quad L = \frac{\pi}{\omega}.$$

Solution: Let the FS is given by, $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E\sin\omega t\,dt = \frac{E}{\pi} \quad a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cos n\omega t\,dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt.$$

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E}{2\pi}\left(\frac{-\cos(1+n)\omega t + 1}{(1+n)} + \frac{-\cos(1-n)\omega t + 1}{(1-n)}\right)$$

$a_1 = 0$, If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \ldots),$$

In a similar manner, $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \ldots\right).$$