Chapter One

FOURIER SERIES

1.2 Half-Range Expansions

Expansion of $f(x)$ given for $0 \le x \le L$ in two Fourier series:

(a) One having only cosine terms known as even series

(b) The other only sine terms known as odd series

Even function: If $f(x)$ is an even function, that is, $f(-x) = f(x)$ its Fourier series reduces to a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad f(x) \text{ even}$$

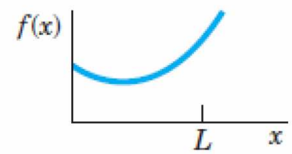
With coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 1, 2, ...$$

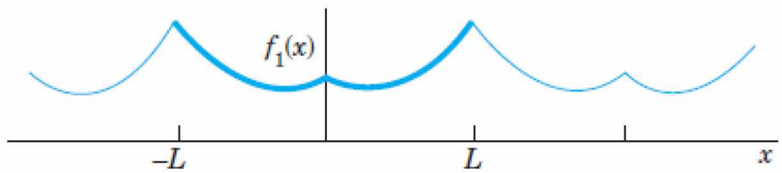
Odd function: If $f(x)$ is an odd function, that is, $f(-x) = -f(x)$, its Fourier series reduces to a **Fourier sine series**

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad f(x) \text{ odd}$$

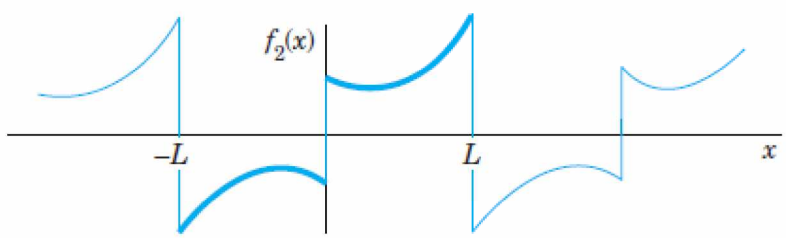
With coefficients $b_n = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}dx$



(0) The given function $f(x)$



(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Fig: Even and odd extensions of period $2L$

Example: "Triangle" and its half-range expansions

Find the two half-range expansions of the function

$$f(x)=\begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0<x<\dfrac{L}{2} \\ \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2}<x<L, \end{cases}$$

Solution: (a) **Even periodic extension** $f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x$

$$a_0=\frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx+\frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right]=\frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right] = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus, $a_2 = -16k/(2^2\pi^2)$, $a_6 = -16k/(6^2\pi^2)$, $a_{10} = -16k/(10^2\pi^2)$, ...,

and $a_n = 0\, if\ n \neq 2,\ 6,\ 10,\ 14,\ ....$

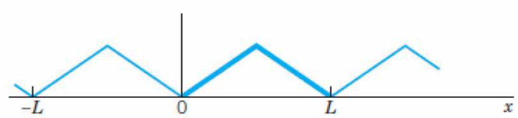
Hence, the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}6....\right).$$

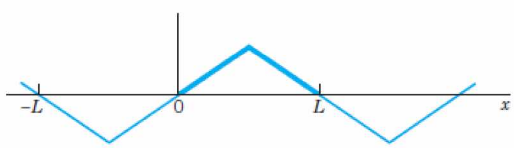
(b) Odd periodic extension. Similarly, $b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$.

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - ...\right).$$



(a) Even extension



(b) Odd extension