Tata Institute of Fundamental Research (GS 2020)

(TOPIC: QUANTUM MECHANICS)

SECTION A

(For both Int. Ph.D. and Ph.D. candidates)

Correct answer will get +3 marks, an incorrect answer will get –1 mark.

Q18. The wave function of a particle subjected to a three-dimensional spherically-symmetric potential $V(r)$ is given by

$$\psi(\vec{x}) = (x + y + 3z) f(r)$$

the expectation value for the operator $\vec{L}^2$ for this state is

(a) $\hbar^2$ (b) $2\hbar^2$ (c) $5\hbar^2$ (d) $11\hbar^2$

Ans.: (b)

Solution: $\psi(\vec{x}) = (x + y + 3z) f(r)$

$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$\psi(r,\theta,\phi) = (r\sin\theta(\cos\phi + \sin\phi) + 3r\cos\theta) f(r)$

which can be written in basis of $p_l(\cos\theta)\exp im\phi$ for $l = 1$ so expectation value will $2\hbar^2$

Q19. A fermion of mass m, moving in two dimensions, is strictly confined inside a square box of side ℓ. The potential inside is zero. A measurement of the energy of the fermion yields the result

$$E = \frac{65\pi^2\hbar^2}{2m\ell^2}$$

The degeneracy of this energy state is

(a) 2 (b) 4 (c) 8 (d) 16

Ans.: (c)

Solution: For fermion one particle will adjust in one quantum state .

So $n_x^2 + n_y^2 = 65$ where n_x and n_y must be integer .

The possible collection $(n_x, n_y) = (1,8),(8,1),(4,7),(7,4)$ are the 4 quantum state

if we consider spin of fermion is $\frac{1}{2}$ then the degeneracy is $\left(2\times\frac{1}{2}+1\right)\times 4 = 8$

if we consider spin of fermion is $\frac{3}{2}$ then the degeneracy is $\left(2\times\frac{3}{2}+1\right)\times 4 = 16$ in this problem

hence spin of fermions is not given then so both can be possible answer .

Q21. The momentum operator

$$i\hbar\frac{d}{dx}$$

acts on a wavefunction $\psi(x)$. This operator is Hermitian

(a) provided the wavefunction $\psi(x)$ is normalized

(b) provided the wavefunction $\psi(x)$ and derivate $\psi'(x)$ are continuous everywhere

(c) provided the wavefunction $\psi(x)$ vanishes as $x \to \pm\infty$

(d) by its very definition

Ans. (c)

Solution: Since p_x is Hermitian

$$\int_{-\infty}^{\infty} \psi^* i\hbar \frac{d\psi}{dx} dx \int_{-\infty}^{\infty} -i\hbar \frac{d\psi^*}{dx}\psi dx = \int_{-\infty}^{\infty} \psi^* i\hbar \frac{d\psi}{dx} dx + \int_{-\infty}^{\infty} i\hbar \frac{d\psi^*}{dx}\psi dx \Rightarrow ih\int_{-\infty}^{\infty} \frac{d\psi^*\psi}{dx} dx = 0$$

$$\int_{-\infty}^{\infty} \frac{d\psi^*\psi}{dx}.dx = 0 \Rightarrow \int_{-\infty}^{\infty} d|\psi|^2 = 0 \Rightarrow |\psi|^2\Big|_{-\infty}^{\infty} = 0 \ \psi(x) \text{ vanishes as } x \to \pm\infty$$

SECTION B

(Only for Int.-Ph.D. candidates)

Correct answer will get +5 marks, an incorrect answer will get 0 mark.

Q34. A particle of mass m is confined inside a box with boundaries at $x = \pm L$. The ground state and the first excited state of this particle are E_1 and E_2 respectively.

Now a repulsive delta function potential $\lambda\delta(x)$ is introduced at the centre of the box where the constant λ satisfies

$$0 < \lambda \ll \frac{1}{32m}\left(\frac{h}{L}\right)^2$$

If the energies of the new ground state and the new first excited state be denoted as E_1' and E_2' respectively, it follows that

(a) $E_1' > E_1, E_2' > E_2$ (b) $E_1' = E_1, E_2' = E_2$

(c) $E_1' > E_1, E_2' = E_2$ (d) $E_1' = E_1, E_2' > E_2$

Ans. : (c)

Solution: $E_1 = \frac{\pi^2\hbar^2}{8mL^2}, E_2 = \frac{4\pi^2\hbar^2}{8mL^2}$

After introduction of

$\lambda\delta(x)$ with $0 < \lambda \ll \frac{1}{32m}\left(\frac{h}{L}\right)^2 \Rightarrow 0 < \lambda \ll \frac{(2\pi)^2}{32m}\left(\frac{\hbar}{L}\right)^2 \Rightarrow 0 < \lambda \ll \frac{\pi^2\hbar^2}{8mL^2}$

which will be more than $E_1' > E_1 = \frac{\pi^2\hbar^2}{8mL^2}$

and other bound state will not be disturbed, so $E_1' > E_1, E_2' = E_2$

Q35. Three noninteracting particles whose masses are in the ratio $1:4:16$ are placed together in the same harmonic oscillator potential $V(x)$.

The degeneracies of the first three energy eigenstates (ordered by increasing energy) will be

(a) $1,1,1$ (b) $1,1,2$ (c) $1,2,1$ (d) $1,2,2$

Ans. : (b)

Solution: The system will analogically behave like three dimensional harmonic oscillator .

Let us assume $\omega_1 = \sqrt{\frac{k}{m}} = \omega, \omega_2 = \sqrt{\frac{k}{4m}} = \frac{\omega}{2}\ \omega_3 = \sqrt{\frac{k}{16m}} = \frac{\omega}{4}$

The energy is $E=\left(n_1+\frac{1}{2}\right)\hbar\omega_2+\left(n_2+\frac{1}{2}\right)\hbar\omega_2+\left(n_3+\frac{1}{2}\right)\hbar\omega_3$

Ground state is $n_1=0,n_2=0,n_3=0$ which is non degenerate $E_{0.0.0}=\frac{\hbar\omega}{2}+\frac{\hbar\omega}{4}+\frac{\hbar\omega}{8}=\frac{7\hbar\omega}{8}$ which is non degenerate

First excited is $n_1=0,n_2=0,n_3=1$ which is non degenerate $E_{0.0.1}=\frac{\hbar\omega}{2}+\frac{\hbar\omega}{4}+\frac{3\hbar\omega}{8}=\frac{9\hbar\omega}{8}$ which is non degenerate

Second excited is $n_1=0,n_2=0,n_3=2$ which is non degenerate $E_{0.0.2}=\frac{\hbar\omega}{2}+\frac{\hbar\omega}{4}+\frac{5\hbar\omega}{8}=\frac{11\hbar\omega}{8}$

other combination is also $n_1=0,n_2=1,n_3=0$ $E_{0.0.2}=\frac{\hbar\omega}{2}+\frac{3\hbar\omega}{4}+\frac{\hbar\omega}{8}=\frac{11\hbar\omega}{8}$ so it is double degenerate

So degeneracy for ground first and second excited state is $1,1,2$.

SECTION B

(Only for Ph.D. candidates)

Correct answer will get +5 marks, an incorrect answer will get 0 mark.

Q34. A particle of mass m is placed in one dimensional harmonic oscillator potential

$$V(x)=\frac{1}{2}m\omega^2x^2$$

At $t=0$, its wavefunction is $\psi(x)$. At $t=2\pi/\omega$ its wavefunction will be

(a) $\psi(x)$ (b) $-\psi(x)$ (c) $-\pi\psi(x)$ (d) $\frac{2\pi}{\omega}\psi(x)$

Ans. : (b)

Solution: Assume $\phi_n(x)$ is energy eigen state with energy eigen value $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$ where

So $\psi\left(x,t=0\right)=\sum_{n=0}c_n\phi_n(x)$

$\psi\left(x,t=t\right)=\sum_{n=0}c_n\phi_n(x)\exp(-\frac{iE_nt}{\hbar})$ for $t=2\pi/\omega$

$$\psi\left(x,t=\frac{2\pi}{\omega}\right)=\sum_{n=0}c_n\phi_n(x)\exp\left(-i\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{\hbar}.\frac{2\pi}{\omega}\right)$$

$$\psi\left(x,t=\frac{2\pi}{\omega}\right)=\sum_{n=0}c_n\phi_n(x)\exp\left(-i\left(n+\frac{1}{2}\right)2\pi\right)$$ where $n=0,1,2...$

$$\psi\left(x,t=\frac{2\pi}{\omega}\right)=\sum_{n=0}c_n\phi_n(x)\exp\left(-i\left(n+\frac{1}{2}\right)2\pi\right)=\sum_{n=0}c_n\phi_n(x)\exp\left(-i\left(2n\pi+\pi\right)\right)$$

$$=-\sum_{n=0}c_n\phi_n(x)=-\psi\left(x\right)$$