Chapter Three

Matrices

3.1 Matrices

Outline

1. Types of matrices, name, and properties
2. Different operations of matrices (Addition and Multiplication)
3. Eigen values and Eigenvectors
4. Functional Matrices: Diagonalization and Similarity transformation

Definition: Matrices are rectangular arrays of numbers or functions.

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $A = \begin{bmatrix} e^x & \sin x \\ x & e^{-x^2} \end{bmatrix}$

Applications:

- It can carry a very large amount of data.
- Scientific and Mathematical ideas can be expressed explicitly.

- Handy for storing and processing the data.
- Very useful in Quantum Mechanics

Representation

$$\begin{bmatrix} & \\ & \end{bmatrix}_{m\times n} \Rightarrow m = Rows,\ n = Columns$$

$$\begin{bmatrix} a_{11} & b_{12} \\ c_{21} & d_{22} \end{bmatrix}_{2\times 2} \Rightarrow Rows = 2,\ Columns = 2$$

3.2 Operations of Matrices

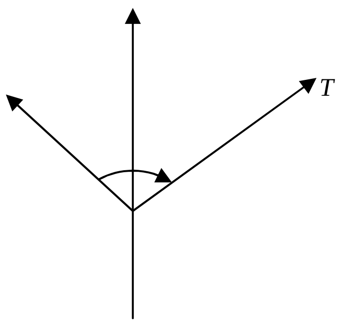
(a) Addition and Subtraction(Only between matrices of equal dimensions)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \pm \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (a\pm 1) & (b\pm 2) \\ (c\pm 3) & (d\pm 4) \end{bmatrix}$$

(b) Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{\substack{2\times 2\\(m\times n)}} \begin{bmatrix} x \\ y \end{bmatrix}_{\substack{2\times 1\\(n\times p)}} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}_{\substack{2\times 1\\(m\times p)}} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

This cab be seen as an example of Linear Transformation.



Here, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transforming matrix

being applied to a vector $\begin{bmatrix} x \\ y \end{bmatrix}$.The origin should be inthe same position.

Example: Find the multiplication of matrices A and B.

(a) $A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}_{3\times 3}$ $and\ \ B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}_{3\times 4}$

Solution: $AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}_{3\times 4}$

$$=\begin{bmatrix} 6+25-9 & -6+4 & 9+35-1 & 3+40-1 \\ 8+18 & -8-8 & 12+2 & 4+2 \\ -12-15+18 & 12-8 & -18-21+2 & -6-24+2 \end{bmatrix}$$

$$AB=\begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

(b) $A=\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}$ *and* $B=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Solution: $AB=\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $A=\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1\times 3}$ & $B=\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}$

Solution: $AB=\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1\times 3}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}=\begin{bmatrix} 19 \end{bmatrix}_{1\times 1}$

(d) $A=\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}$ *and* $B=\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1\times 3}$

Solution: $AB=\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}_{1\times 3}=\begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}_{3\times 3}$

(e) $A=\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ *and* $B=\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}$

$$AB=\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}=\begin{bmatrix} 99 & 99 \\ -101 & -101 \end{bmatrix}$$

$$BA=\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}=\begin{bmatrix} -2 & 0 \\ -200 & 0 \end{bmatrix}$$

$\Rightarrow AB \neq BA$

3.3 Name of matrices and their meaning

1. Square Matrices:

Number of Rows = Number of Columns

Example: $P = \begin{bmatrix} E_1 & E_2 & E_3 \\ E_1' & E_2' & E_3' \\ E_1'' & E_2'' & E_3'' \end{bmatrix}_{3\times 3}$, Here, $m = n$

A^T **(Transpose):**

Let $A = \begin{bmatrix} E_1 & E_2 & E_3 \\ E_4 & E_5 & E_6 \\ E_7 & E_8 & E_9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} E_1 & E_4 & E_7 \\ E_2 & E_5 & E_8 \\ E_3 & E_6 & E_9 \end{bmatrix}$

We can write a matrix as: $A = \left[a_{ij}\right]_{m\times n}$

For example: $a_{11} = E_1$

$a_{12} = E_2$

$a_{31} = E_7$ etc.

(a) $\left(A^T\right)^T = A$

(b) $(AB)^T = B^T A^T$

(c) $(A+B)^T = A^T + B^T$

2. Symmetric Matrix: $\left(A^T\right) = A$

$$A=\begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}, \quad A^T = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} = A$$

$\therefore A$ is a symmetric matrix.

3. Skew Symmetric Matrix: $\left(A^T\right) = -A$

$$B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}, \; B^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

$\because B^T = -B$

$\therefore B$ is a skew-symmetric matrix.

In a skew-symmetric matrix, the diagonal elements will always be equal to zero.

Proof: $A=\left[a_{ij}\right]_{n\times n}$

$$A^T=-A$$

$$\left[a_{ji}\right]_{n\times n}+\left[a_{ij}\right]_{n\times n}=0$$

$$\left[a_{ji}\right]+\left[a_{ij}\right]=0$$

for diagonal elements, $i=j$,

$$\therefore\left[a_{ii}\right]+\left[a_{ii}\right]=0\Rightarrow 2\left[a_{ij}\right]=0$$

$$\Rightarrow\ \left[a_{ij}\right]=0$$

So, diagonal elements of the skew-symmetric matrix are always zero.

4. Orthogonal Matrix: $A^T A=I\ \Rightarrow A^T=A^{-1}$

Also the sum of the square of elements of any row $=1$.

If $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is orthogonal $\Rightarrow \begin{matrix} a^2+b^2+c^2=1 \\ d^2+e^2+f^2=1 \\ g^2+h^2+i^2=1 \end{matrix}$

$$A=\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$$

$$B=\frac{1}{2}\left(A+A^T\right),C=\frac{1}{2}\left(A-A^T\right)\quad\Rightarrow A=B+C$$

Now, $B^T=\frac{1}{2}\left(A^T+A\right)=B$, $\quad\therefore B$ is symmetric

$C^T=\frac{1}{2}\left(A^T-A\right)=-C$, $\quad\therefore C$ is skew-symmetric

So A matrix can be represented in its symmetric and skew-symmetric parts.

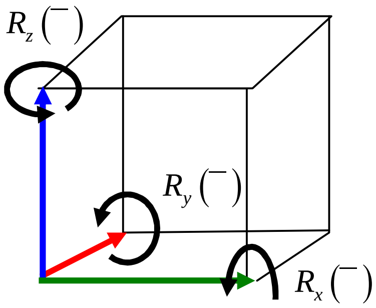
Question: If $P=\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} a & -2b \end{bmatrix}$, then what will be the symmetric part?

Solution : $P=\begin{bmatrix} a \\ b \end{bmatrix}\begin{bmatrix} a & -2b \end{bmatrix}=\begin{bmatrix} a^2 & -2ab \\ ba & -2b^2 \end{bmatrix}$

For symmetric part $B=\frac{1}{2}\left(P+P^T\right)=\frac{1}{2}\left\{\begin{bmatrix} a^2 & -2ab \\ ba & -2b^2 \end{bmatrix}+\begin{bmatrix} a^2 & ab \\ -2ab & -2b^2 \end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix} 2a^2 & -ab \\ -ab & -4b^2 \end{bmatrix}$

$B=\begin{bmatrix} a^2 & -ab/2 \\ -ab/2 & -2b^2 \end{bmatrix}=$Symmetric part of P.

5. Hermitian Matrix: $A^\dagger = A$



Example: $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

Take transpose $\Rightarrow A^T = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$,

Then take the conjugate $\Rightarrow A^\dagger = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$\Rightarrow A^\dagger = A \therefore A \rightarrow$ Hermitian Matrix

6. Skew HermititionMatrix : $A^\dagger = -A$

Example: $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$A^\dagger = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix} = -\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}, \quad A^\dagger = -A$

$\therefore A \rightarrow$ Skew Hermitian matrix

7. Unitary Matrix: $A^\dagger A = I$ or $A^\dagger = A^{-1}$

8. Triangular Matrix: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ = Principal diagonal

If all elements lie below the principal diagonal are zero $\Rightarrow$ Upper triangular matrix

And if all elements lie above principle diagonal are zero $\Rightarrow$ Lower triangular Matrix

9. Diagonal Matrix:

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

At least one of the Diagonal terms should be non-zero. (We are talking about principle diagonal). Also, all other elements are zero.

10. Unit matrix:

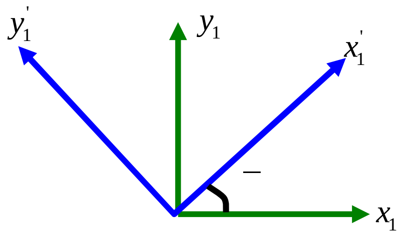
Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Principle diagonal has 1's and all other elements are zero. Also known as Identity matrix.

11. Rotation Matrix in 2D:

$$A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\begin{bmatrix}x_1'\\ y_1'\end{bmatrix}$$



Rotation Matrix in 3D:

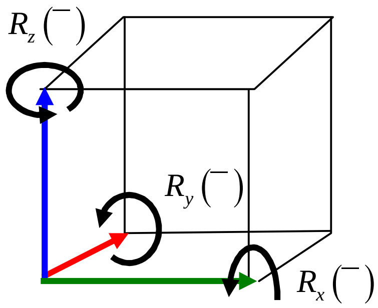
$+\theta \rightarrow$ For anticlockwise direction

$-\theta \rightarrow$ For clockwise direction

$$A_x(\pm\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \mp\sin\theta\\ 0 & \pm\sin\theta & \cos\theta\end{bmatrix}$$

$$A_y(\pm\theta)=\begin{bmatrix}\cos\theta & 0 & \pm\sin\theta\\ 0 & 1 & 0\\ \mp\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$A_z(\pm\theta)=\begin{bmatrix}\cos\theta & \mp\sin\theta & 0\\ \pm\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$



3.4 Elementary Row Transformation

Uses:

(a) Inverse

(b) Linear Equation

(c) Rank

Allowed Operation:

(a) Interchange of rows

(b) Addition of a constant multiple of one row to other

(c) Multiplication of an equation by a non-zero constant.

$$\Rightarrow 3x_1+2x_2+2x_3-5x_4=8$$
$$0.6x_1+1.5x_2+1.5x_3-5.4x_4=2.7$$
$$1.2x_1-0.2x_2-0.3x_3+2.4x_4=2.1$$

$$\Rightarrow Let\ A=\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.2 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1 & -1.1 & 4.4 & -1.1 \end{bmatrix}, \begin{pmatrix} R_2 \to R_2 - 0.2R_1 \\ R_3 \to R_3 - 0.4R_1 \end{pmatrix}$$

$$\Rightarrow A=\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0.6 & 0.1 & 0 & 0 & 0 \end{bmatrix}, (R_3 \to R_3 + R_2)$$

3.5 Linear Dependence and Independence

$$\vec{A}=[A_1 \quad A_2 \quad A_3]$$

$$\vec{B}=[B_1 \quad B_2 \quad B_3]$$

Let vectors be

$a_{(1)}, a_{(2)}, a_{(3)}, \ldots\ldots\ldots a_{(m)}$ (A)

Linear combination:- $c_1 a_{(1)} + c_2 a_{(2)} + c_3 a_{(3)} + \ldots\ldots\ldots c_{(m)} a_{(m)}$

Now Let,

$c_1 a_{(1)} + c_2 a_{(2)} + c_3 a_{(3)} + \ldots\ldots\ldots + c_{(m)} a_{(m)} = 0$ (1)

(a) Equation (1) is only true if all $c_m\text{'}s = 0$ then vectors in equation (A) are Linearly Independent.

(b) Ex: $c_1 a_{(1)} + c_2 a_{(2)} = 0 \Rightarrow a_{(2)} = \frac{-c_1 a_{(1)}}{c_2}$, So Linearly Dependent.

$\because a_{(2)}$ can be written in terms of $a_{(1)}$,that's why they are linearly dependent.

Linearly independent vectors are truly essential vectors.

Example: Let $a_{(1)} = \{3 \quad 0 \quad 2 \quad 2\}$

$a_{(2)} = \{-6 \quad 42 \quad 24 \quad 54\}$

$a_{(3)} = \{21 \quad -21 \quad 0 \quad -15\}$

$\Rightarrow 6a_{(1)} - \frac{1}{2} a_{(2)} - a_{(3)} = 0 \ (\because c_1, c_2, c_3 \neq 0)$ So $a_{(1)}, a_{(2)} \ \& \ a_{(3)}$ are Linearly dependent.

And $a_{(1)}, a_{(2)}$ are Linearly Independent.

3.6 The Rank of the Matrix

The rank of a matrix A is the maximum number of linearly independent row vectors A.

$$\text{Let } A=\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$\therefore$ Rank $=2$ because the first two rows are linearly independent.

Now by **Echelon form:**

$$A=\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$A=\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}, \begin{pmatrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{pmatrix}$$

$$A=\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left(R_3 \to R_3 + \frac{1}{2}R_2\right)$$

$\therefore$ Rank 2

$\Rightarrow A$ and A^T have the same rank.

Suppose we have n number of vectors and all vectors are Linearly Independent.

Then, rank $= n =$ the dimension of the matrix.

$\therefore$ Rank can be either $= n$ (for Linearly Independent) or

Less than n (for Linearly Dependent)

3.7 The Inverse of a Matrix

(a) Square matrices $(n\times n)$: $A, A^{-1} \Rightarrow A^{-1}A = I$

$A^{-1}AA^{-1} = A^{-1}$

$A^{-1}I = A^{-1}$

So, if we have $A = AI$

$I = AA^{-1}$

(b) $A^{-1} = \dfrac{adj\ A}{|A|} \quad \Rightarrow |A| \neq 0$

$adj(A) = (\text{cofactor } A)^T$

3.8 Eigenvalues and Eigenvectors

Square matrix $A_{n\times n}X_{n\times 1} = \lambda X_{n\times 1}$, Where λ is eigenvalue.

$\Rightarrow (A-\lambda I)X = 0$

Taking determinant $\Rightarrow |A-\lambda I| X = 0$

Now, $f(\lambda)=0$ $\Rightarrow$ The Characteristic equation

Example: If A is 3×3 matrix, $f(\lambda)$ will be a cubic equation.

$\lambda_1, \lambda_2, \lambda_3$ are eigenvalues.

$\Rightarrow$ Every matrix follows its own characteristic equation

And $X_1, X_2, X_3 \rightarrow$ Eigenvectors corresponds to $\lambda_1, \lambda_2, \lambda_3$.

Example: If $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$, Then what will be eigenvalues, inverse, and eigenvectors of the given matrix A?

Solution: For eigenvalues:

$$[A-\lambda I]=0 \Rightarrow \begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=0 \Rightarrow \begin{bmatrix}-5-\lambda & 2-0\\ 2-0 & -2-\lambda\end{bmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0 \Rightarrow \lambda^2+7\lambda+6=0$$

$\Rightarrow \lambda_1=-6$ & $\lambda_2=-1$ are the eigenvalues.

By Cayley Hamilton Theorem, $A^2+7A+6I=0$

$\Rightarrow A^2=-[7A+6I]$

For Inverse:

$$A^{-1}(A^2+7A+6I)=0 \Rightarrow A+7I+6A^{-1}=0 \Rightarrow A^{-1}=\frac{-1}{6}[A+7I]$$

$$A^{-1}=-\frac{1}{6}\begin{bmatrix}2 & 2\\ 2 & 5\end{bmatrix}$$

Or

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{\det A}\begin{bmatrix}a_{22} & -a_{21}\\ -a_{12} & a_{11}\end{bmatrix}$$

$$\det A=10-4=6 \Rightarrow A^{-1}=\frac{1}{6}\begin{bmatrix}-2 & -2\\ -2 & -5\end{bmatrix}$$

For Eigenvector:

$$A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix} \quad \Rightarrow a_1=-6\ \&\ a_2=-1$$

Eigenvector:

For $a_1=-6$:

$$[A-aI]X=0 \Rightarrow [A+6I]X=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$\begin{bmatrix}x_1+2x_2\\ 2x_1+4x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \Rightarrow x_1+2x_2=0,\quad 2x_1+4x_2=0$$

$$\Rightarrow x_1=-2x_2$$

Let $x_2=1$, Then $x_1=-2$

So $X_1=\begin{bmatrix}-2\\ 1\end{bmatrix} \Rightarrow$ Normalized form: $X_1=\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ 1\end{bmatrix}$

For $a_2=-1$:

$$[A+I]Y=\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$-4y_1+2y_2=0\ \ \&\ 2y_1-y_2=0 \Rightarrow y_2=2y_1$$

Let $y_1=1$, Then $y_2=2$

So $X_2=\begin{bmatrix}1\\ 2\end{bmatrix} \Rightarrow$ Normalized form: $X_2=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\ 2\end{bmatrix}$

Now $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix} \because A^T=A \Rightarrow$ Symmetric Matrix

$$X_1=\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ 1\end{bmatrix}, X_2=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\ 2\end{bmatrix}\ \&\ X_1^T=\frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\end{bmatrix}$$

$$X_1^T X_2=\frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1\\ 2\end{bmatrix}_{2\times 1}=\frac{1}{5}[-2+2]=0$$

So for a symmetric matrix, the eigenvectors are orthogonal.

Location of Eigenvalues

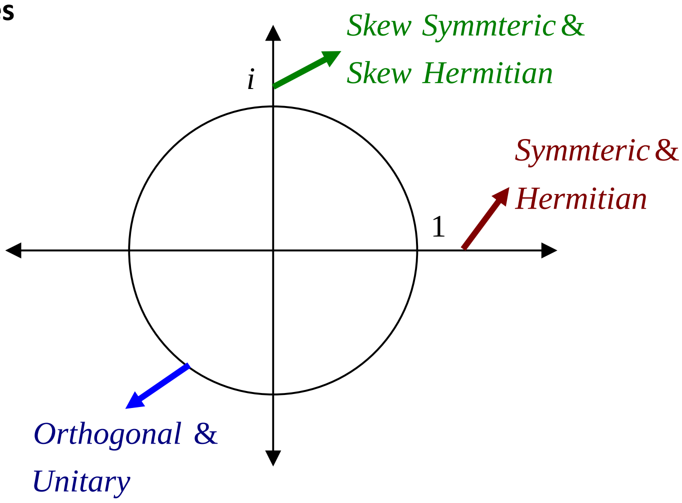


Fig. Argand Plane

Question: $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$, Find the Eigenvalues, Determinant, Trace, and Eigenvector corresponding to eigenvalues.

Solution: $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=0$

$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$

$(\lambda+3)^2(\lambda-5)=0$, Eigenvalues are $\lambda_1=-3, \lambda_2=-3, \lambda_3=5$

$A^3+A^2-21A-45I=0 \Rightarrow A^3+A^2-21A=45I=\begin{bmatrix}45 & 0 & 0\\ 0 & 45 & 0\\ 0 & 0 & 45\end{bmatrix}$

Eigenvalues $=5,-3,-3$

$\Rightarrow$ Trace $(A)=\Sigma\lambda_i=\lambda_1+\lambda_2+\lambda_3=5-3-3=-1$

And Determinant $(A)=\Pi\lambda_i=\lambda_1\times\lambda_2\times\lambda_3=(5)(-3)(-3)=45$

Eigenvector:

For $\lambda=5$

$$[A-5I]X_1=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}7 & 2 & -3\\ 1 & -2 & -3\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\left(R_2\rightarrow\frac{1}{2}R_2\right)$$

$$=\begin{bmatrix}-7 & 2 & -3\\ 1 & -2 & -3\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\left(R_3\to R_3+R_2\ \&\ R_3\to\left(\frac{-1}{4}\right)R_3\right)$$

$$=\begin{bmatrix}0 & 1 & 2\\ 1 & -2 & -3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\left(R_1\to R_1+7R_2\,,R_1\to-\frac{R_1}{12}\right)$$

$x_2+2x_3=0$, $x_1-2x_2-3x_3=0$, $x_2=-2x_3$

Let, $x_3=k, x_2=-2k$

Then $x_1=2x_2+3x_3=-4k+3k=-k$

$$X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=k\begin{bmatrix}-1\\-2\\1\end{bmatrix}\text{ or }k\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

So For $\lambda_1=5\left(when\,k=1\right)\Rightarrow X_1=\begin{bmatrix}1\\2\\-1\end{bmatrix}$, Normalized form: $X_1=\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}$

For degenerate eigenvalue, $\lambda_2=-3$

$$\left|A-\lambda I\right|=0\Rightarrow\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}=0$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow\begin{bmatrix}1 & 2 & -3\\ 1 & 2 & -3\\ -1 & -2 & 3\end{bmatrix},\left(R_2\to\frac{R_2}{2}\right)$$

$$=\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & x_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\left(R_1\to R_1-R_2\ \&\ R_3\to R_3-R_2\right)$$

$x_1+2x_2-3x_3=0$

let $x_1=k_1\ \&\ x_2=k_2\ \Rightarrow x_3=\frac{k_1+2k_2}{3}$

$$X_2=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}k_1\\ k_2\\ \frac{1}{3}\left(k_1+2k_2\right)\end{bmatrix}$$

So For $^{a}{}_{2} = -3\left(when\, k_1 = k_2 = 1\right) \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, Normalized form: $X_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Talking X_3 such that X_2 and X_3 at are Linearly independent.

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

3.9 Pauli Spin Matrices

$$\pm_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \pm_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) $\pm_j \pm_k = i\pm_l$

$(j,k,l) \to (1,2,3),(2,3,1),(3,1,2)$

$\Rightarrow \pm_1\pm_2 = i\pm_3$

(a) $(\pm_i)^2 = ?$

$$(\pm_1)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\pm_2)^2 = \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\pm_3)^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(b) $|\pm_i| = -1$,Orthogonal Matrix

(c) $\pm_i\pm_j + \pm_j\pm_i = 2¢_{ij}I$, $¢_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$

$¢_{ij}$ = Kronecker Delta, $\hat{i} \neq j$

$$\pm_1\pm_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \pm_2\pm_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$\Rightarrow \pm_1\pm_2 + \pm_2\pm_1 = 0$

For $i = j \Rightarrow \pm_1^2 + \pm_2^2 = 2I$

Question: If P & Q is Real Symmetric, then which of the following option is correct for PQ?

(a) Symmetric for all P and Q

(b) Never Symmetric

(c) Symmetric if $PQ = QP$

(d) Anti Symmetric for all P and Q

Answer: (c)

Solution: $(PQ)^T = Q^T P^T = QP$

Question: If $M = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, the eigenvalues of M are?

(a) Real and positive

(b) Purely imaginary with mod 1

(c) Complex with mod 1

(d) Real and negative

Answer: (c)

Solution: $|M - a I| = 0 \quad \Rightarrow \begin{bmatrix} \frac{i}{\sqrt{2}} - a & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} - a \end{bmatrix} = 0$

$$\left(\frac{i}{\sqrt{2}} - a\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0 \qquad \Rightarrow \left(\frac{\hat{i}}{\sqrt{2}} - a - \frac{1}{\sqrt{2}}\right)\left(\frac{\hat{i}}{\sqrt{2}} - a + \frac{1}{\sqrt{2}}\right) = 0$$

$$a_1 = \frac{i+1}{\sqrt{2}}, \qquad a_2 = \frac{i-1}{\sqrt{2}}$$

$\Rightarrow i^\dagger = -i$ (Conjugate)

Question: Given two $(n \times n)$ matrices such that $\hat{P}$ is Hermitian and $\hat{Q}$ is skew Hermitian Matrix. Then which of the following combination is a Hermitian matrix.

(a) $\hat{P}\hat{Q}$ (b) $i\hat{P}\hat{Q}$ (c) $\hat{P} + i\hat{Q}$ (d) $\hat{P} + \hat{Q}$

Answer: (c)

Solution: (a) $\left(\hat{P}\hat{Q}\right)^\dagger = Q^\dagger P^\dagger = -QP$

(b) $\left(i\hat{P}\hat{Q}\right)^\dagger = i^\dagger Q^\dagger P^\dagger = (-i)(-Q)(P) = iQP$

(c) $\left(\hat{P} + i\hat{Q}\right)^\dagger = P^\dagger + i^\dagger Q^\dagger = P + (-i)(-Q) = \hat{P} + i\hat{Q}$

(d) $\left(\hat{P}+\hat{Q}\right)=P^{\dagger}+Q^{\dagger}=P-Q$

Question: For the three matrices given below, which one of the options is correct?

$$\pm_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\pm_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\pm_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

(a) $\pm_1\pm_2=-i\pm_3$

(b) $\pm_1\pm_2=i\pm_3$

(c) $\pm_1\pm_2+\pm_2\pm_1=I$

(d) $\pm_3\pm_2=-i\pm_1$

Answer: (b)

Solution: $\pm_1\pm_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}=i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$\pm_1\pm_2=i\pm_3$

Question: If $M=\begin{bmatrix}4 & x\\ 6 & 9\end{bmatrix}$ and $\det M=0$. Then which of the following option is correct for Matrix M?

(a) M is symmetric

(b) M is invertible

(c) One eigenvalue is 13

(d) Its eigenvectors are orthogonal

Answer: (a)

Solution: $M=\begin{bmatrix}4 & x\\ 6 & 9\end{bmatrix}\Rightarrow\det M=\begin{vmatrix}4 & x\\ 6 & 9\end{vmatrix}=36-6x=0\ \Rightarrow x=6$

$M=\begin{bmatrix}4 & 6\\ 6 & 9\end{bmatrix}$

For eigenvalue:

$$\begin{bmatrix}4-a & 6\\ 6 & 9-a\end{bmatrix}=0\quad\Rightarrow(4-a)(9-a)-36=0$$

$\Rightarrow a^2-13a=0\ \ So\ \ a_1=13, a_2=0$

For eigenvector:

$$A-13I=\begin{bmatrix}-9 & 6\\ 6 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$-9x_1+6x_2=0,\ 6x_1-4x_2=0 \Rightarrow -3x_1+2x_2=0$

Let $x_1,=1$ then $x_2=\frac{3}{2} \Rightarrow X_1=\begin{bmatrix}1\\3/2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$

Question: $M=\begin{bmatrix}3 & i & 0\\-i & 3 & 0\\0 & 0 & 6\end{bmatrix}$, What will be its eigenvalue?

(a) $2,4,6$ (b) $2i,4i,6$ (c) $2i,4,8$ (d) $0,4,8$

Answer: (a)

Solution: $T_r(A)=3+3+6=12$

$|M|=3(18)-i(-6i)+0=54-6=48$

And determinant is the product of eigenvalues.

3.10 Similarity Transformation

Eigen basis $\rightarrow$ Diagonal matrix

$A\rightarrow n\times n$ (Given), $\hat{A}\rightarrow n\times n$ (Similar matrix)

$\hat{A}=P^{-1}AP \Rightarrow \hat{A}$ is similar to A

$\hat{A}=P^{-1}AP$: This transformation is known as similarity transformation.

- It preserves the eigenvalues of A.
- If x is an eigenvector of A, then $y=p^{-1}x$ is an eigenvector of $\hat{A}$ corresponding to the same eigenvalue.
- A matrix P can be reflected such that the similarity transformation results into a D. (diagonal matrix)

3.11 Diagonalization

If a $n \times n$ matrix A has a basis of eigenvectors then $D = X^{-1}AX$,is a diagonal matrix, with eigenvalues of A as the entries on the main diagonal. (principal diagonal) and X is the matrix with eigenvectors as column vectors.

$$D = X^{-1}AX$$

Question: Find the eigenvalues and diagonalized form of the matrix $A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$.

Solution: $A - aI = \begin{bmatrix} 7.3 - a & 0.2 & -3.7 \\ -11.5 & 1 - a & 5.5 \\ 17.7 & 1.8 & -9.3 - a \end{bmatrix}$

$$= (7.3 - a)\left[a^2 + 8.3a - 19.2\right] - 1.92 - 2.a + 142.08 - 65.49a$$

$$= a^3 + a^2 - 12a = 0 \quad \Rightarrow a\left(a^2 + a - 12\right) = 0$$

$$a_1 = -4, a_2 = 3, a_3 = 0$$

For eigenvector:

$$a_1 = 3:$$

$$A - 3I = \begin{bmatrix} 4.3 & 0.2 & -3.7 \\ -11.5 & -2 & 5.5 \\ 17.7 & 1.8 & -12.3 \end{bmatrix}$$

$$(A - 3I)X = \begin{bmatrix} 4.3 & 0.2 & -3.7 \\ -11.5 & -2 & 5.5 \\ 17.7 & 1.8 & -12.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4.3x_1 + 0.2x_2 - 3.7x_3 = 0$ (1)

$-11.5x_1 - 2x_2 + 5.5x_3 = 0$ (2)

$17.7x_1 + 1.8x_2 - 12.3x_3 = 0$ (3)

After solving the above equations we will get,

$$\Rightarrow x_1 = x_3$$

$$x_2 = -3x_1$$

Let $x_1 = -1, x_2 = 3, x_3 = -1$

$$\Rightarrow \alpha_1 = 3 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

For $\alpha_2 = -4$:

$$[A+4I]X_2 = \begin{bmatrix} 11.3 & 0.2 & -3.7 \\ -11.5 & 5 & 5.5 \\ 17.7 & 1.8 & -5.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$11.3x_1 + 0.2x_2 - 3.7x_3 = 0$$

$$-11.5x_1 + 5x_2 + 5.5x_3 = 0$$

$$17.7x_1 + 1.8x_2 - 5.3x_3 = 0$$

After solving the above three equations we will get

$$x_3 = 3x_1 \quad \& \quad x_1 = -x_2$$

Let $x_1 = 1$, $\Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

Now for $\alpha_3 = 0$

$$[A-0]X_3 = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7.3x_1 + 0.2x_2 - 3.7x_3 = 0$$

$$-11.5x_1 + x_2 + 5.5x_3 = 0$$

$$17.7x_1 + 1.8x_2 - 9.3x_3 = 0$$

By solving the above equations:

$$x_1 = 2x_2,\ x_3 = 4x_2$$

Let $x_2 = 1 \quad \Rightarrow Then\ x_1 = 2 \Rightarrow x_3 = 4$

$$\therefore X_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$X = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \Rightarrow |X| = 7 - 1(13) + 2(8) = 10$$

$$X^{-1}=\frac{adjX}{|X|}$$

$$adjX=\begin{bmatrix} \begin{vmatrix}-1 & 1\\ 3 & 4\end{vmatrix} & -\begin{vmatrix}3 & 1\\ -1 & 4\end{vmatrix} & \begin{vmatrix}3 & -1\\ -1 & 3\end{vmatrix}\\ -\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} & \begin{vmatrix}-1 & 2\\ -1 & 4\end{vmatrix} & -\begin{vmatrix}-1 & 1\\ -1 & 3\end{vmatrix}\\ \begin{vmatrix}-1 & 2\\ -1 & 1\end{vmatrix} & -\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix} & \begin{vmatrix}-1 & 1\\ 3 & -1\end{vmatrix}\end{bmatrix}$$

$$adjX=\begin{bmatrix}-7 & -13 & 8\\ 2 & -2 & 2\\ 1 & 7 & -2\end{bmatrix}^T=\begin{bmatrix}-7 & 2 & 1\\ -13 & -2 & 7\\ 8 & 2 & -2\end{bmatrix}$$

$$X^{-1}=\frac{adjX}{|X|}=\begin{bmatrix}-0.7 & 0.2 & 0.1\\ -1.3 & -0.2 & 0.7\\ 0.8 & 0.2 & -0.2\end{bmatrix}$$

$$D=X^{-1}AX=\begin{bmatrix}3 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Important property:

$$D=X^{-1}AX$$

- $A=XDX^{-1}$
- $A^n=XD^nX^{-1}$

Example: $A^{100}=XD^{100}X^{-1}$

$$D=\begin{bmatrix}a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{bmatrix} \quad \Rightarrow D^{100}=\begin{bmatrix}a_1^{100} & 0 & 0\\ 0 & a_2^{100} & 0\\ 0 & 0 & a_3^{100}\end{bmatrix}$$

$$\Rightarrow D=\begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix} \quad \Rightarrow D^{100}=\begin{bmatrix}a_1^{100} & 0\\ 0 & a_2^{100}\end{bmatrix}$$

$$e^D=\begin{bmatrix}e^{a_1} & 0\\ 0 & e^{a_2}\end{bmatrix} \quad \Rightarrow \cos D=\begin{bmatrix}\cos a_1 & 0\\ 0 & \cos a_2\end{bmatrix}$$

- The inner product is always normalized.

3.12 Functional Matrices

If A is a $n \times n$ matrix then,

(a) $\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

(b) $\sin(A) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} A^{2j+1} = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots$

(c) $\cos(A) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} A^{2j} = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots$

3.13 Spectral Decomposition Law

Once the eigenvalues and eigenvectors of a Hermitian matrix H have been found, the expression we now derive for H is referred to as its spectral decomposition.

$$H = \sum_i f(\lambda_i) |r_i\rangle\langle r_i|$$

Each r_i satisfies $Hr_i = f(\lambda_i) r_i$ and $\langle r_i | r_i \rangle = 1$.

Another result related to the spectral decomposition of H can be obtained if we multiply both sides of the equation $Hr_i = \lambda_i r_i$ on the left by H, reaching

$$H^2 r_i = \lambda_i r_i$$

Further applications of H show that all positive powers of H have the same eigenvectors as H, so if $f(H)$ is any function of H that has a power-series expansion, it has the spectral decomposition $f(H) = \sum_i f(\lambda_i) |r_i\rangle\langle r_i|$

The above equation can be extended to include negative powers if H is nonsingular; to do so, multiply $Hr_i = \lambda_i r_i$ on the left by H^{-1} and rearrange, to obtain $H^{-1} r_i = \frac{1}{\lambda_i} r_i$ showing that negative powers of H also have the same eigenvectors as H.

Example: $A_{3\times 3} \Rightarrow |r_1\rangle_{3\times 1} \langle r_1|_{1\times 3}$

$$f(A_{3\times 3}) = f(\lambda_1)|r_1\rangle\langle r_1| + f(\lambda_2)|r_2\rangle\langle r_2| + f(\lambda_3)|r_3\rangle\langle r_3|$$

Question: $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, Then find e^{σ_3}.

Solution: Eigenvalues : $(1-a)(-1-a)=0$

$$a=\pm 1,\ a_1=1, a_2=-1$$

Eigenvector:

$$a_1=1\rightarrow r_1=\begin{bmatrix}1\\0\end{bmatrix},\quad a_2=-1\rightarrow r_2=\begin{bmatrix}0\\1\end{bmatrix}$$

$$e^{\pm_3}=e^{1}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}+e^{-1}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$e^{\pm_3}=\begin{bmatrix}e & 0\\ 0 & \frac{1}{e}\end{bmatrix}$$

Example: Consider a matrix $M=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

(a) Eigenvalues

(b) $e^{M}=?$

Solution: (a) $\begin{bmatrix}1-a & 1 & 1\\ 1 & 1-a & 1\\ 1 & 1 & 1-a\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$(1-a)\left[(1-a)^2-1\right]-1\left[(1-a)-1\right]+1\left[1-1+a\right]=0$$

$$a_1=0, a_2=0, a_3=3$$

(b) $e^{M}=I+\frac{M}{1!}+\frac{M^2}{2!}+\cdots\cdots$

$$M^2=3M, M^3=3^2M, M^4=3^3M$$

$$e^{M}=I+M+\frac{3M}{2!}+\frac{3^2M}{3!}+\cdots\cdots=I+M\left[1+\frac{3}{2!}+\frac{3^2}{3!}+\cdots\cdots\right]$$

$$e^{M}=I+M\left[\frac{1}{3}\left(1+\frac{3}{1!}+\frac{3^2}{2!}+\cdots\cdots\right)-\frac{1}{3}\right]$$

$$e^{M}=I+\left(\frac{e^3-1}{3}\right)^{M}$$

Question: A 3×3 matrix M has $Tr(M)=6, Tr(M^2)=26, Tr(M^3)=90$

Which of the following options can be the possible set of eigenvalues?

(a) $1,1,4$ (b) $-1,3,4$ (c) $-1,0,7$ (d) $2,2,2$

Answer: (b)

Solution: $T_r[M]=-1+3+4=6$

$$T_r[M^2]=1^2+3^2+4^2=16+9+1=26$$

$$T_r[M^3]=(-1)^3+3^3+4^3=-1+27+64=90$$

Question: The eigenvalees of anti-symmetric matrix $A=\begin{bmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{bmatrix}$

Where n_1,n_2,n_3 are the components of a unit vector. Then its eigenvalues will be:

(a) $0,-i,i$ (b) $0,1,-1$ (c) $0,1,+i,1-i$ (d) $0,0,0$

Answer: (a)

Solution: $|A-\lambda I|=\begin{bmatrix}0-\lambda & -n_3 & n_2\\ n_3 & 0-\lambda & -n_1\\ -n_2 & n_1 & 0-\lambda\end{bmatrix}$

$$=(-\lambda)\left[\lambda^2+n_1^2\right]+n_3\left[-\lambda n_3-n_1n_2\right]+n_2\left[n_1n_3-\lambda n_2\right]=0$$

$$=-\lambda^3-\lambda n_1^2-\lambda n_3^2-n_1n_2n_3+n_1n_2n_3-\lambda n_2^2=0$$

$$=-\lambda^3-\lambda\left[n_1^2+n_2^2+n_3^2\right]=0$$

$$-\lambda\left[\lambda^2+1\right]=0 \qquad \lambda=0,\pm i$$

And also since the matrix is antisymmetric so eigenvalues should be either zero or pure imaginary.

Question: Given a 2×2 Unitary matrix U satisfying $U^\dagger U=I=UU^\dagger$, with determinant $U=e^{i\theta}$, construct a unitary matrix $V\left(V^+V=VV^+=1\right)$ with determinant $V=1$ from it by

(a) Multiplying U by $e^{-i\theta/2}$

(b) Multiplying any single element of U by $e^{-i\theta}$

(c) Multiplying any row or column of U by $e^{-i\theta/2}$

(d) Multiplying U by $e^{-i\theta}$

Answer: (a)

Solution: $det(\sim U) = \sim^n \det U, \quad n = 2$

$1 = \sim^2 e^{i¤} \Rightarrow \sim^2 = e^{-i¤}, \ \sim = e^{-i¤/2}$

Question: Consider the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The condition for the existence of a non-trivial solution and the corresponding normalized solution is.

Solution: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{bmatrix}$

$[4c - 3b] - 1[2c - 6] + 1[b - 4] = 0$

$4c - 3b - 2c + 6 + b - 4 = 0$

$2c - 2b + 2 = 0 \qquad 2c = 2b - 2 \Rightarrow \quad b = c + 1$

Question: Consider the three vectors $\vec{v}_1 = 2\hat{i} + 3\hat{k},\ \vec{v}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{v}_3 = 5\hat{i} + \hat{j} + a\hat{k}$. These vectors will be linearly dependent if the value of a is:

Solution: $[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 3 & 2 & a \end{bmatrix} \Rightarrow 2(2a-2) - 1(-3) + 5(-6) = 0$

$4a - 4 + 3 - 30 = 0 \qquad 4a - 1 - 30 = 0$

$4a - 31 = 0 \qquad a = \frac{31}{4}$

Special Unitary Matrix:

$$SU(2) = \begin{Bmatrix} \sim & - \\ & \sim \end{Bmatrix}$$

$|\sim|^2 + |\quad|^2 = 1$

Example: $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$\Rightarrow a^2 - a(Tr(A)) + \det A = 0$

$a^3 - Tr(A)a^2 + (a_{11} + a_{22} + a_{33})a - \det A = 0.$