Laplace Transformation

1. Definition of Laplace Transform

- This is one of the integral transformations.
- It is used to solve Ordinary and Partial Differential Equation. (Initial value problem and boundary value problem).

Laplace Transform

$$h(t) \Leftrightarrow H(f)$$

Inverse Laplace Transform

Further deriving function in the form of a unit step function, Heaviside function, and Dirac delta function becomes very simple in Laplace transformation.

IVP —(*L.T.*)→ *Algebraic Problem* —(*I.L.T.*)→ *Solution* where, IVP: Initial value problem

L.T.: Laplace transforms and I.L.T.: Inverse Laplace transforms

Applications:

- Electrical Networks
- Spring
- Mixing problem
- Signal Processing
- Many other areas of engineering and physics.

First principle of Laplace Transformation:

$$F(s)=\int_0^\infty e^{-st} f(t)dt$$

- $F(s) \Leftrightarrow f(t)$: They will form Laplace transformation pairs.
- $f(t)=L^{-1}F(s)$ and $F(s)=Lf(t)$

Example: Find the Laplace transform of the following functions:

(a) $f(t)=1$

Solution: $F(s)=\int_0^\infty e^{-st} f(t)\,dt$

$$F(s)=\int_0^\infty e^{-st}(1)\,dt=\int_0^\infty \left[e^{-st}\right]dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=\frac{-1}{s}\left[e^{-\infty}-e^{0}\right]=\frac{1}{s}$$

$$L\{1\}=\frac{1}{s}$$

(b) $f(t)=e^{at}$

Solution: $F(s)=\int_0^\infty e^{at}\cdot e^{-st}dt=\left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty=\frac{1}{a-s}\left[e^{(a-s)\infty}-e^{0}\right]$

If $s>a$

$$F(s)=\frac{1}{a-s}[0-1]=\frac{1}{s-a}$$

Laplace holds the linear property:

$$L\left[af(t)+bf(t)\right]=aLf(t)+bLf(t)$$

(c) $f(t)=\sinh at$

Solution: $L(\sinh at)=L\left\{\frac{e^{at}-e^{-at}}{2}\right\}=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]$

$$=\frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]=\frac{a}{s^2-a^2}$$

$$L(\sinh at)=\frac{a}{s^2-a^2}$$

(d) $f(t)=\cosh at$

Solution: $L\{\cosh at\}=L\left\{\frac{e^{at}+e^{at}}{2}\right\}=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]=\frac{s}{s^2-a^2}$$

$$L(\cosh at)=\frac{s}{s^2-a^2}$$

(e) $f(t)=\sin\omega t$

Solution: $L\{\sin\omega t\}=L\left\{\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right\}$

$$=\frac{1}{2i}\left[L\left(e^{i\omega t}\right)-L\left(e^{-i\omega t}\right)\right]=\frac{1}{2i}\left(\frac{1}{s-i\omega}-\frac{1}{s+i\omega}\right)=\frac{1}{2i}\left(\frac{s+i\omega-s+i\omega}{s^2+\omega^2}\right)=\frac{2i\omega}{2i\left(s^2+\omega^2\right)}$$

$$L\{\sin\omega t\}=\frac{\omega}{\left(s^2+\omega^2\right)}$$

(f) $f(t)=\cos\omega t$

Solution: $L\{\cos\omega t\}=L\left\{\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right\}=\frac{1}{2}\left[Le^{i\omega t}+Le^{-i\omega t}\right]$

$$=\frac{1}{2}\left[\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right]=\frac{1}{2}\left(\frac{s+i\omega+s-i\omega}{s^2+\omega^2}\right)=\frac{s}{s^2+\omega^2}$$

$$L(\cos\omega t)=\frac{s}{s^2+\omega^2}$$

(g) $f(t)=t^{n+1}$

Solution: $L\{t^{n+1}\}=\int_0^\infty e^{-st}t^{n+1}dt$

Let $st=z \Rightarrow (t=z/s)$

$sdt=dz$

$$=\int_0^\infty e^{-z}\frac{z^{n+1}}{s^{n-1}}dz=\frac{1}{s^{n+2}}\int_0^\infty e^{-z}z^{n+1}dz$$

$$L\{t^{n+1}\}=\frac{(n+1)!}{s^{n+2}}$$

Shifting property:

If $L[f(t)]=F(s)$

$$L[e^{at}f(t)]=F(s-a)$$

Example: If $L(\cos\omega t)=\frac{s}{s^2+\omega^2}$, Then $L\{e^{at}\cos(\omega t)\}=\frac{s-a}{(s-a)^2+\omega^2}$

And if $L(\sin\omega t)=\frac{\omega}{s^2+\omega^2}$, Then $L\{e^{at}\sin(\omega t)\}=\frac{\omega}{(s-a)^2+\omega^2}$

Find the Laplace transform of the following functions:

(a) $f(s)=\frac{3s-137}{s^2+2s+401}$

Solution: $f(s)=\frac{3s-137}{s^2+2s+401}=\frac{3s-137}{s^2+2s+1+400}$

$$=\frac{3s-137}{(s+1)^2+(20)^2}=\frac{3(s+1)-140}{(s+1)^2+(20)^2}$$

$$=3\frac{(s+1)}{(s+1)^2+(20)^2}-7\frac{20}{(s+1)^2+(20)^2}$$

$$f(t)=L^{-1}\{F(s)\}=3L^{-1}\left\{\frac{(s+1)}{(s+1)^2+(20)^2}\right\}-7L^{-1}\left\{\frac{20}{(s+1)^2+(20)^2}\right\}$$

$$f(t)=3(e^{-t}\cos 20t)-7(e^{-t}\sin 20t)$$

$$f(t)=e^{-t}(3\cos 20t-7\sin 20t)$$

(b) $f(t)=3t+12$

Solution: $L\{(t)\}=3L\{t\}+12L\{1\}=\frac{3}{s^2}+\frac{12}{s}$

(c) $f(t)=(a-bt)^2$

Solution: $L\{f(t)\}=a^2L\{1\}+b^2L\{t^2\}-2abL\{t\}$

$$=\frac{a^2}{s}+b^2\frac{2!}{s^3}-2ab\frac{1!}{s^2}$$

$$=\frac{a^2}{s}+\frac{2b^2}{s^3}-\frac{2ab}{s^2}$$

(d) $f(t)=\cos^2\omega t'$

Solution: $f(t)=\cos^2\omega t$

$$f(t)=\frac{1+\cos 2\omega t}{2}$$

$$L(f(t))=\frac{1}{2}L\{1\}+\frac{1}{2}L\{\cos 2\omega t\}=\frac{1}{2}\frac{1}{s^2}+\frac{1}{2}\frac{s}{(s^2+4\omega^2)}$$

(e) $f(t)=\sin(\omega t+\theta)$

Solution: $f(t)=\sin(\omega t+\theta)=\sin\omega t\cos\theta+\cos\omega t\sin\theta$

$$L\{f(t)\}=\cos\theta L\{\sin\omega t\}+\sin\theta L\{\cos\omega t\}$$

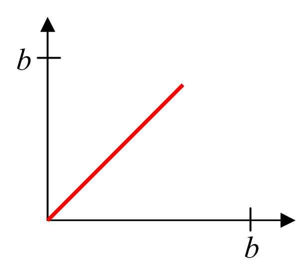
$$=\frac{\omega\cos\theta}{s^2+\omega^2}+\frac{\sin\theta s}{s^2+\omega^2}$$

(f) $f(t)=\begin{cases}k & 0<t<c\\ 0 & t>c\end{cases}$

Solution: $F(s)=\int_0^\infty f(t)e^{-st}=\int_0^c ke^{-st}dt=k\left[\frac{e^{-st}}{-s}\right]_0^c=\frac{-k}{s}\left[e^{-sc}-1\right]$

$$F(s)=\frac{k}{s}\left[1-e^{-sc}\right]$$

(g) $f(t)=\left\{\begin{matrix}t & 0\le t\le b\\ 0 & t>b\end{matrix}\right\}$

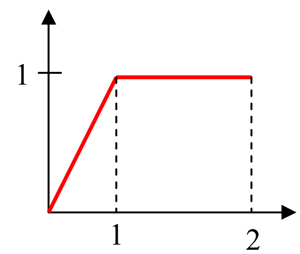


Solution: $F(s)=\int_0^{\infty} e^{-st} f(t)\,dt=\int_0^{b} e^{-st}t\,dt$

$$=\left[\frac{te^{-st}}{-s}-\frac{1e^{-st}}{s^2}\right]_0^b$$

$$F(s)=\frac{be^{-sb}}{-s}-\frac{e^{-sb}}{s^2}+\frac{1}{s^2}=\frac{e^{-sb}}{s^2}\left[e^{sb}-1+bs\right]$$

(h) $f(t)=\begin{Bmatrix} t & 0\le t\le 1 \\ 1 & 2<t<1 \\ 0 & t>2 \end{Bmatrix}$

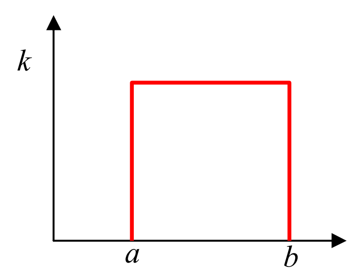


Solution: $F(s)=\int_0^{\infty} e^{-st} f(t)\,dt=\int_0^{1} e^{-st}t\,dt+\int_1^{2} e^{-st}\,dt$

$$=\left[\frac{te^{-st}}{-s}-\frac{1e^{-st}}{s^2}\right]_0^1+\left[\frac{e^{-st}}{-s}\right]_1^2$$

$$F(s)=\frac{e^{-2s}}{s^2}\left[e^{2s}-e^{s}-s\right]$$

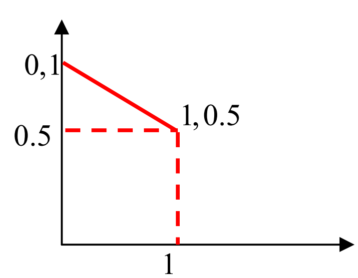
(i) $f(t)=k \qquad a<t<b$



Solution: $F(s)=Lf(t)=\int_0^{\infty} f(t)e^{-st}\,dt=\int_a^{b} k\,\mathrm{e}^{-st}\,dt=k\left(\frac{e^{-st}}{-s}\right)_a^b$

$$=k\left(\frac{e^{-sa}-e^{-sb}}{s}\right)$$

(j) $f(t)=\left(1-\frac{t}{2}\right) \qquad 0<t<1$



Solution: $F(s)=Lf(t)=\int_0^{\infty} f(t)e^{-st}\,dt=\int_0^{1}\left(1-\frac{t}{2}\right)\mathrm{e}^{-st}\,dt$

$$=\left(\frac{e^{-st}}{-s}\right)_0^1-\frac{1}{2}\left(\frac{te^{-sa}}{-s}-\frac{e^{-st}}{s^2}\right)_0^1$$

$$=\frac{e^{-s}}{-s}+\frac{1}{s}-\frac{1}{2}\left(\frac{e^{-s}}{-s}-0-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)$$

$$F(s)=\frac{1}{s}-\frac{1}{2}\frac{e^{-s}}{s}+\frac{1}{2}\frac{e^{-s}}{s^2}-\frac{1}{2}\frac{1}{s^2}$$