

## 1.8 The Spherical Polar Coordinate System

Important points of spherical polar coordinate system

(a) Let us start by building a spherical coordinate system on the three rectangular axes (Figure 1.10(a)).

(b) We first define the distance from the origin to any point as  $r$ . The surface  $r = \text{constant}$  is a sphere. The second coordinate is an angle  $\theta$  between the  $z$  axis and the line drawn from the origin to the point in question. The surface  $\theta = \text{constant}$  is a cone, and the two surfaces, cone and sphere, are everywhere perpendicular along their intersection, which is a circle of radius  $r \sin \theta$  (Fig. 1.10(b)).

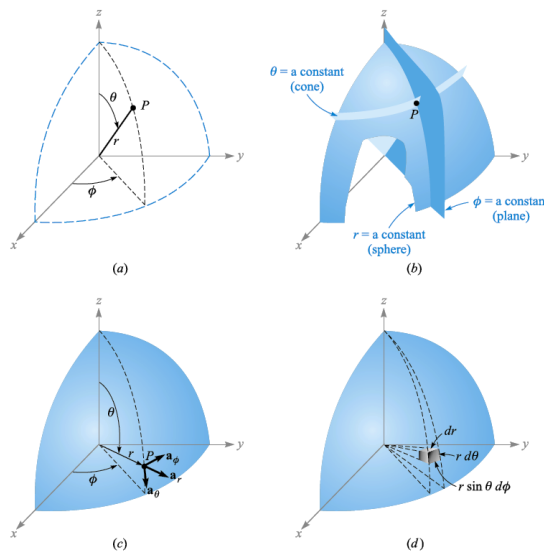


Fig. 1.10 (a) The three spherical coordinates. (b) The three mutually perpendicular surfaces of the spherical coordinate system. (c) The three unit vectors of spherical coordinates:  $a_r \times a_\theta = a_\phi$ . (d) The differential volume element in the spherical coordinate system.

(c) The third coordinate  $\phi$  is also an angle and is exactly the same as the angle  $\phi$  of cylindrical coordinates. It is the angle between the  $x$  axis and the projection in the  $z = 0$  plane of the line drawn from the origin to the point.

(d) Consider any point as the intersection of three mutual surfaces—a sphere, a cone, and a plane—each oriented in the same manner. The three surfaces are shown in Figure 1.10(b)

**Note:** Three unit vectors may again be defined at any point. Each unit vector is perpendicular to one of the three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases.

### The unit vector

(i)  $\mathbf{a}_r$  is directed radially outward, normal to the sphere  $r = \text{constant}$ , and lies in the cone  $\theta = \text{constant}$  and the plane  $\phi = \text{constant}$ .

(ii) The unit vector  $\mathbf{a}_\theta$  is normal to the conical surface, lies in the plane, and is tangent to the sphere. It is directed along a line of "longitude" and points "south."

(iii) The third unit vector  $\mathbf{a}_\phi$  is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and the sphere. It is directed to the "east."

(iv) The three unit vectors are shown in Figure 1.10(c). They are, of course, mutually perpendicular, and a right-handed coordinate system is defined by causing  $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$ . Our system is right-handed, as an inspection of Figure 1.10(c) will show, on application of the definition of the cross product. The right-hand rule identifies the thumb, forefinger, and middle finger with the direction of increasing  $r$ ,  $\theta$ , and  $\phi$ , respectively. (Note that the identification in cylindrical coordinates was with  $\rho$ ,  $\phi$ , and  $z$ , and in rectangular coordinates with  $x$ ,  $y$ , and  $z$ .)

(v) A differential volume element may be constructed in spherical coordinates by increasing  $r$ ,  $\theta$ , and  $\phi$  by  $dr$ ,  $d\theta$ , and  $d\phi$ , as shown in Figure 1.10(d). The distance between the two spherical surfaces of radius  $r$  and  $r + dr$  is  $dr$ ; the distance between the two cones having generating angles of  $\theta$  and  $\theta + d\theta$  is  $r d\theta$ ; and the distance between the two radial planes at angles  $\phi$  and  $\phi + d\phi$  is found to be  $r \sin \theta d\phi$ , after a few moments of trigonometric thought.

**Note:** The surfaces have areas of  $r dr d\theta$ ,  $r \sin \theta dr d\phi$ , and  $r^2 \sin \theta d\theta d\phi$ , and the volume is  $r^2 \sin \theta dr d\theta d\phi$

The transformation of scalars from the rectangular to the spherical coordinate system is easily made by using Figure 1.10(a) to relate the two sets of variables:

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

The transformation in the reverse direction is achieved with the help of

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} & (r \geq 0) \\ \theta &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} & (0^\circ \leq \theta \leq 180^\circ) \\ \phi &= \tan^{-1} \frac{y}{x} \end{aligned}$$

The radius variable  $r$  is non-negative, and  $\theta$  is restricted to the range from  $0^\circ$  to  $180^\circ$ , inclusive. The angles are placed in the proper quadrants by inspecting the signs of  $x$ ,  $y$ , and  $z$ .

**Transformation of Vectors:** Table Dot products of unit vectors in spherical and rectangular coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin \theta \cos \phi$	$\cos \theta \cos \phi$	$-\sin \phi$
$\mathbf{a}_y \cdot$	$\sin \theta \sin \phi$	$\cos \theta \sin \phi$	$\cos \phi$
$\mathbf{a}_z \cdot$	$\cos \theta$	$-\sin \theta$	$0$

**Example:** We illustrate this procedure by transforming the vector field  $\mathbf{G} = (xz/y)\mathbf{a}_x$  into spherical components and variables.

**Solution:** We find the three spherical components by dotting  $\mathbf{G}$  with the appropriate unit vectors, and we change variables during the procedure:

$$\begin{aligned}
 G_r &= \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y} \sin \theta \cos \phi \\
 &= r \sin \theta \cos \theta \frac{\cos^2 \phi}{\sin \phi} \\
 G_\theta &= \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y} \cos \theta \cos \phi \\
 &= r \cos^2 \theta \frac{\cos^2 \phi}{\sin \phi} \\
 G_\phi &= \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y} \mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y} (-\sin \phi) \\
 &= -r \cos \theta \cos \phi
 \end{aligned}$$

Collecting these results, we have

$$\mathbf{G} = r \cos \theta \cos \phi (\sin \theta \cot \phi \mathbf{a}_r + \cos \theta \cot \phi \mathbf{a}_\theta - \mathbf{a}_\phi)$$

## Examples

**Example 1:** Specify the unit vector extending from the origin toward the point  $G(2,-2,-1)$ .

**Solution:** We first construct the vector extending from origin to point  $G$ ,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - 1\mathbf{a}_z$$

We continue by finding the magnitude of  $\mathbf{G}$ ,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \frac{2}{3}\mathbf{a}_x - \frac{2}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are  $\mathbf{u}_B$ ,  $\mathbf{a}_B$ ,  $\mathbf{1}_B$ , or even  $\mathbf{b}$ . We will consistently use the lowercase  $\mathbf{a}$  with an appropriate subscript.

**Example 2:** Given Points  $M(-1, 2, 1)$ ,  $N(3, -3, 0)$ , and  $P(-2, -3, -4)$ , find: (a)  $\mathbf{R}_{MN}$ ; (b)  $\mathbf{R}_{MN} + \mathbf{R}_{MP}$ ; (c)  $|\mathbf{r}_M|$ ; (d)  $\mathbf{a}_{MP}$ ; (e)  $|2\mathbf{r}_P - 3\mathbf{r}_N|$ .

**Solution:** (a)  $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$ ; (b)  $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$ ; (c) 2.45; (d)  $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$ ; (e) 15.56

**Example 3:** A vector field  $\mathbf{S}$  is expressed in rectangular coordinates as  $\mathbf{S} = \left\{ 125 / \left[ (x-1)^2 + (y-2)^2 + (z+1)^2 \right] \right\} \{ (x-1)\mathbf{a}_x + (y-2)\mathbf{a}_y + (z+1)\mathbf{a}_z \}$ . (a) Evaluate  $\mathbf{S}$  at  $P(2, 4, 3)$ . (b) Determine a unit vector that gives the direction of  $\mathbf{S}$  at  $P$ . (c) Specify the surface  $f(x, y, z)$  on which  $|\mathbf{S}| = 1$ .

**Solution:** (a)  $5.95\mathbf{a}_x + 11.90\mathbf{a}_y + 23.8\mathbf{a}_z$ ; (b)  $0.218\mathbf{a}_x + 0.436\mathbf{a}_y + 0.873\mathbf{a}_z$ ; (c)  $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$

**Example 4:** In order to illustrate these definitions and operations, consider the vector field  $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$  and the point  $Q(4, 5, 2)$ . We wish to find:  $\mathbf{G}$  at  $Q$ ; the scalar component of  $\mathbf{G}$  at  $Q$  in the direction of  $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$ ; the vector component of  $\mathbf{G}$  at  $Q$  in the direction of  $\mathbf{a}_N$ ; and finally, the angle  $\theta_{Ga}$  between  $\mathbf{G}(\mathbf{r}_Q)$  and  $\mathbf{a}_N$ .

**Solution:** Substituting the coordinates of point  $Q$  into the expression for  $\mathbf{G}$ , we have

$$\mathbf{G}(\mathbf{r}_Q) = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\mathbf{G} \cdot \mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z) \cdot \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = \frac{1}{3}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of  $\mathbf{a}_N$

$$(\mathbf{G} \cdot \mathbf{a}_N) \mathbf{a}_N = -(2) \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

The angle between  $\mathbf{G}(\mathbf{r}_Q)$  and  $\mathbf{a}_N$  is found from

$$\begin{aligned} \mathbf{G} \cdot \mathbf{a}_N &= |\mathbf{G}| \cos \theta_{Ga} \\ -2 &= \sqrt{25 + 100 + 9} \cos \theta_{Ga} \end{aligned}$$

and

$$\theta_{Ga} = \cos^{-1} \frac{-2}{\sqrt{134}} = 99.9^\circ$$

**Example 5:** The three vertices of a triangle are located at  $A(6, -1, 2)$ ,  $B(-2, 3, -4)$ , and  $C(-3, 1, 5)$ . Find: (a)  $\mathbf{R}_{AB}$ ; (b)  $\mathbf{R}_{AC}$ ; (c) the angle  $\theta_{BAC}$  at vertex  $A$ ; (d) the (vector) projection of  $\mathbf{R}_{AB}$  on  $\mathbf{R}_{AC}$ .

**Solution:** (a)  $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$ ; (b)  $-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ ; (c)  $53.6^\circ$ ; (d)  $-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

**Example 6:** The three vertices of a triangle are located at  $A(6, -1, 2)$ ,  $B(-2, 3, -4)$ , and  $C(-3, 1, 5)$ . Find: (a)  $\mathbf{R}_{AB} \times \mathbf{R}_{AC}$ ; (b) the area of the triangle; (c) a unit vector perpendicular to the plane in which the triangle is located.

**Solution:** (a)  $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z$ ; (b) 42.0; (c)  $0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

**Example 7:** Calculate the volume of the parallelepiped formed by following the three vectors

$$\mathbf{I} = \mathbf{a}_x + \mathbf{a}_y, \mathbf{J} = 4\mathbf{a}_y + \mathbf{a}_z, \text{ and } \mathbf{K} = 2\mathbf{a}_y + 2\mathbf{a}_z$$

All the length in above vectors may be considered to be in cm.

**Solution:** Consider the parallelepiped shown in Figure 1.7, formed by three vectors  $\mathbf{I}$ ,  $\mathbf{J}$ , and  $\mathbf{K}$ . The volume of the parallelepiped can actually be determined by estimating the  $STP \equiv \mathbf{I} \cdot (\mathbf{J} \times \mathbf{K})$  as described earlier. Now, in order to determine the  $STP$ , the inner vector product may first be estimated using the following expression:

$$(\mathbf{J} \times \mathbf{K}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ J_x & J_y & J_z \\ K_x & K_y & K_z \end{vmatrix} = \mathbf{a}_x (J_y K_z - J_z K_y) + \mathbf{a}_y (J_z K_x - J_x K_z) + \mathbf{a}_z (J_x K_y - J_y K_x)$$

Once the above vector product is determined, the resultant  $STP$  may be computed as follows:

$$\mathbf{I} \cdot (\mathbf{J} \times \mathbf{K}) \equiv I_x (J_y K_z - J_z K_y) + I_y (J_z K_x - J_x K_z) + I_z (J_x K_y - J_y K_x)$$

Finally, the values of different components may be substituted in the above equation to determine the total volume. The component values of all the three vectors in the present example are given as follows:

$$I_x = 1, I_y = 1, I_z = 0, J_x = 0, J_y = 4, J_z = 1, K_x = 0, K_y = 2, K_z = 2$$

Hence, the actual volume is estimated as Volume of parallelepiped ( $STP$ ) =  $(8 - 2) + 0 + 0 = 6 \text{ cm}^3$

**Example 8:** Estimate the VTP  $\mathbf{I} \times (\mathbf{J} \times \mathbf{K})$  corresponding to the vectors given in the previous example and comment about the direction of the resultant vector.

**Solution:**

$$\begin{aligned} \text{VTP} \equiv \mathbf{I} \times (\mathbf{J} \times \mathbf{K}) &= \mathbf{J}(\mathbf{I} \cdot \mathbf{K}) - \mathbf{K}(\mathbf{I} \cdot \mathbf{J}) \\ &= \mathbf{J} \left[ (\mathbf{a}_x + \mathbf{a}_y) \cdot (2\mathbf{a}_y + 2\mathbf{a}_z) \right] - \mathbf{K} \left[ (\mathbf{a}_x + \mathbf{a}_y) \cdot (4\mathbf{a}_y + \mathbf{a}_z) \right] \\ &= 2(4\mathbf{a}_y + \mathbf{a}_z) - 4(2\mathbf{a}_y + 2\mathbf{a}_z) \\ &= -6\mathbf{a}_z \end{aligned}$$

**Example 9:** Transform the vector  $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$  into cylindrical coordinates.

**Solution:** The new components are

$$\begin{aligned} B_\rho &= \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho) \\ &= y \cos \phi - x \sin \phi = \rho \sin \phi \cos \phi - \rho \cos \phi \sin \phi = 0 \end{aligned}$$

$$\begin{aligned} B_\phi &= \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi) \\ &= -y \sin \phi - x \cos \phi = -\rho \sin^2 \phi - \rho \cos^2 \phi = -\rho \end{aligned}$$

Thus,

$$\mathbf{B} = -\rho \mathbf{a}_\phi + z \mathbf{a}_z$$