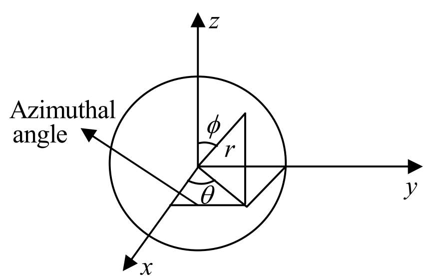
Multiple Variable Calculus

3. Cartesian to Spherical Polar Coordinates

Limits are difficult

$$\iiint dxdydz \rightarrow \int_0^{\pi}\int_0^{2\pi}\int_0^{R} J(r,\theta,\phi)\,drd\theta d\phi$$



$x = r\cos\theta\sin\theta$

$y = r\sin\theta\sin\theta$

$z = r\cos\theta$

$$J(r,\theta,\phi) = \begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

$$= \cos\theta\sin\phi\left[-r\cos\theta\sin^2\phi - 0\right] + r\sin\theta\sin\phi\left[-r\sin\theta\sin^2\phi - r\sin\theta\cos^2\phi\right]$$

$+r\cos\theta\cos\phi\left[-r\cos\theta\sin\phi\cos\phi\right]$

$J=-r^2\sin\theta$

$|J|=r^2\sin\theta$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2\sin\theta dr d\theta d\phi=\frac{4}{3}\pi R^3$$

Example: Find the Jacobian for the following transformation

$x=2u+3v-\omega$

$y=v-5\omega$

$z=u+4\omega$

Solution: $J(u,v,\omega)=\begin{vmatrix}2 & 3 & -1\\ 0 & 1 & -5\\ 1 & 0 & 4\end{vmatrix}$

$=8-14$

$J(u,v,\omega)=-6$

Example: $x=u^2+v\omega$

$y=2v+u^2\omega$

$z=uv+\omega$

$$J(u,v,\omega)=\begin{vmatrix}2u & \omega & v\\ 2u\omega & 2 & u^2\\ v\omega & u\omega & uv\end{vmatrix}$$

$=2u\left(2uv-u^3\omega\right)-\omega\left(2u^2v\omega-u^2v\omega\right)+v\left(2u^2\omega^2-2v\omega\right)$

$=4u^2v-2u^4\omega-u^2v\omega^2+2u^2\omega^2v-2v^2\omega$

$=4u^2v-2u^4\omega+2u^2\omega^2v-2v^2\omega$

Example: $x=p\cos\theta$

$y=p\sin\theta$

$z=z$

Solution: $J(\rho,\theta,z)=\begin{vmatrix}\cos\theta & -\rho\sin\theta & 0\\ \sin\theta & \rho\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}$

$$=\cos\theta[\rho\cos\theta]+\rho\sin\theta[\sin\theta]=\rho\cos^2\theta+\rho\sin^2\theta$$

$$J=\rho$$

Example: $H(q_1,p_1;q_2,p_2)=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+k\left(q_1^2+q_2^2+\frac{q_1+q_2}{4}\right);k>0$

$$Z(\beta)=\int_{-\infty}^{+\infty}dq_1\int_{-\infty}^{+\infty}dp_1\int_{-\infty}^{+\infty}dq_2\int_{-\infty}^{+\infty}dp_2 e^{-\beta H}$$

$$e^{-\beta H}=e^{-\beta\frac{p_1^2}{2m}}\cdot e^{-\beta\frac{p^2}{2m}}\cdot e^{-\beta k\left(q_1^2+q_2^2+\frac{1}{4}q_1q_2\right)}$$

$$\Rightarrow Z(\beta)=\int_{-\infty}^{+\infty}e^{-\beta\frac{p_1^2}{2m}}dp_1\int_{-\infty}^{+\infty}e^{-\beta\frac{p_2^2}{2m}}dp_2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-\beta k\left(q_1^2+q_2^2+\frac{q_1q_2}{4}\right)}dq_1dq_2$$

$$=\sqrt{\frac{\pi}{\beta/2m}}\sqrt{\frac{\pi}{\beta/2m}},$$

How to manage this?

Let, $q_1=u+v\Rightarrow u=(q_1+q_2)/2$

$$q_2=u-v\Rightarrow v=(q_1-q_2)/2$$

$$J(u,v)=\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=-1-1=-2$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|J|e^{-\beta k(..)}dudv$$

$$q_1^2+q_2^2+\frac{1}{4}q_1q_2=\frac{9}{4}u^2+\frac{7}{4}v^2$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}e^{-\beta k\frac{9}{4}u^2}du\int_{-\infty}^{+\infty}e^{-\beta k\frac{7}{4}v^2}dv=2\sqrt{\frac{\pi}{\beta k/(9/4)}}\sqrt{\frac{\pi}{\beta k/(7/4)}}$$