# Fourier Transformation

**3. Fourier Transformation Properties:**

- **Linearity Property**: $x(t) \Leftrightarrow X(f)$

  And $y(t) \Leftrightarrow Y(f)$

  Linearity property says: $x(t)+y(t) \Leftrightarrow X(f)+Y(f)$

  i.e. $\Rightarrow \int_{-\infty}^{\infty}(x(t)+y(t))e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt + \int_{-\infty}^{\infty}y(t)e^{-j2\pi ft}dt$

  $$= X(f)+Y(f)$$

- **Symmetry property**: If $h(t) \Leftrightarrow H(f)$ (If $h(t)$ and $H(f)$ make Fourier transform)

  Then symmetry property says:

  $$h(t) \Leftrightarrow H(f)$$

  $$\& \quad H(t) \Leftrightarrow h(-f)$$

**Example:** $e^{-t^2/2\sigma^2} \Leftrightarrow \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$

Where $h(t) = e^{-t^2/2\sigma^2}$ & $H(f) = \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$

Then $H(t) \Leftrightarrow h(-f)$

$$\Rightarrow \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 t^2} \Leftrightarrow e^{-f^2/2\sigma^2}$$

**Example:** Given Fourier transformation of $e^{-t^2/2\sigma^2}$ is $\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$. Find the Fourier transform $\sigma\sqrt{2\pi}e^{-2\pi^2\sigma^2 t^2}$ using symmetry property.

**Solution:** Using symmetry property we get $h(-f) = e^{-f^2/2\sigma^2}$

$\Rightarrow$ Prove that $H(t) \Leftrightarrow h(-f)$

**Proof:** Given $H(f) \Leftrightarrow \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt$

And $h(t) \Leftrightarrow \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df$

$$h(-t) \Leftrightarrow \int_{-\infty}^{\infty} H(f) e^{-j2\pi ft} df$$

Switch $t \to f$

$$h(-f) = \int_{-\infty}^{\infty} H(t) e^{j2\pi(-f)t} dt$$

- **Time scaling:** Suppose we have $h(t) \Rightarrow H(f)$

Then $h(kt) \Leftrightarrow \frac{1}{k} H\left(\frac{f}{k}\right)$, where k is real.

We know that, $F(h(t)) = \int_{-\infty}^{\infty} h(kt) e^{-j2\pi ft} dt$

Consider $kt = y$, Then $F(h(t)) = \int_{-\infty}^{\infty} h(y) e^{-j2\pi fy/k} dy$

$$= \frac{1}{k}\int_{-\infty}^{\infty} h(t) e^{-j2\pi ft/k} dt \qquad (y \to t)$$

$$F(h(t)) = \frac{1}{k} \quad H\left(\frac{f}{k}\right)$$

**Example:** If $e^{-t^2/2\sigma^2} \Leftrightarrow \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$

Where $h(t) = e^{-t^2/2\sigma^2}$ & $H(f) = \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$

Then $e^{-t^2/2\sigma^2} \Leftrightarrow \frac{1}{k}\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2\frac{f^2}{k^2}}$

- **Frequency Scaling:** If $h(t) \Leftrightarrow H(f)$

$\frac{1}{|k|}h\left(\frac{t}{k}\right) \Leftrightarrow H(k)$

$\frac{1}{|k|}h\left(\frac{t}{k}\right)$, This will always be positive and k should be real.

- **Time-shifting:** Suppose $h(t) \Leftrightarrow H(f)$

$h(t-t_0) \Leftrightarrow ?$

We know $H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$

And $H(f) = \int_{-\infty}^{\infty} h(t)e^{j2\pi ft}dy$

$F\{h(t-t_0)\} = \int_{-\infty}^{\infty} h(t-t_0)e^{-j2\pi ft}dt$

Now Let $t - t_0 = y$

$= \int_{-\infty}^{\infty} h(y)e^{-j2\pi f(t_0+y)}dy = e^{-j2\pi ft_0}\int_{-\infty}^{\infty} h(y)e^{-j2\pi fy}dy$

Here $y \leftrightarrow t$

$= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$

$F\{h(t-t_0)\} = e^{-j2\pi ft_0}\ H(f)$

**Example:** Given $h(t) \Leftrightarrow H(f)$ i.e. $e^{-t^2/2\sigma^2} \Leftrightarrow \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$, Then find $h(t-\alpha)$?

**Solution:** $h(t-\alpha) = e^{-j2\pi f\alpha}\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 f^2}$

- **Frequency shifting:** If $h(t) \Leftrightarrow H(f)$

Then $H(f-f_0) \Leftrightarrow ?$

We know that $F(h(t)) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$

And $F^{-1}\left[H(f)\right]=\int_{\infty}^{\infty} H(f)e^{j2\pi ft}dt = h(t)$

Let $f - f_0 = y \Rightarrow df = dy$

$$= \int_{-\infty}^{\infty} H(y)e^{j2\pi(f_o+y)}dy$$

$$F^{-1}\left[H(f-f_0)\right] = e^{j2\pi f_0 t}h(t)$$

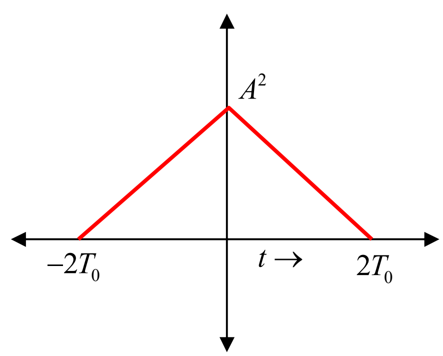
$\Rightarrow$ **In general,** If $h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

Then $F\{h(t)\} = H(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T}\right)$

$\Rightarrow$ The Fourier transformation of a sequence of an equal distant impulse function is another sequence delta function of the equal distant pulse.

**Example:** If $h(t) = \frac{2T_0A^2 + A^2t}{2T_0} \qquad -2T_0 \le t \le 0$

$$= \frac{2T_0A^2 - A^2t}{2T_0} \qquad 0 \le t \le 2T_0$$




Then find its Fourier transform.

**Solution:** $F(h(t)) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f_0 t}dt$

$$= \int_{-2T_0}^{0}\left(A^2 + \frac{A^2t}{2T_0}\right)e^{-j2\pi f_0 t}dt + \int_{0}^{2T_0}\left(A^2 - \frac{A^2t}{2T_0}\right)e^{-j2\pi f_0 t}dt$$

$$= A^2\int_{-2T_0}^{0} e^{-j2\pi f_0 t}dt + \frac{A^2}{2T_0}\int_{-2T_0}^{0} te^{-j2\pi f_0 t}dt + A^2\int_{0}^{2T_0} e^{-j2 f_0 t}dt - \frac{A^2}{2T_0}\int_{0}^{2T_0} te^{-j2\pi f_0 t}dt$$

$$= \left\{A^2\left[\frac{e^{-j2\pi f_0 t}}{-j2\pi f_0}\right]_{-2T_0}^{0} + \frac{A^2}{2T_0}\left[t\frac{e^{-j2\pi f_0 t}}{-j2\pi f_0 t} - (1)\frac{e^{-j2\pi f_0 t}}{(-j2\pi f_0 t)^2}\right]_{-2T}^{0}\right\} + \left\{A^2\left[\frac{e^{-j2\pi f_0 t}}{-j2\pi f_0}\right]_{0}^{2T_0} - \frac{A^2}{2T_0}\left[t\frac{e^{-j2\pi f_0 t}}{-j2\pi f_0} - 1\frac{e^{-j2\pi f_0 t}}{(-j2\pi f_0)^2}\right]_{0}^{2T_0}\right\}$$

$$= \frac{A^2}{j2\pi f_0}\left[e^{j4\pi f_0 T_0} - e^{-j4\pi f_0 T_0} - 1 + 1\right] + \frac{A^2}{2T_0(-j2\pi f_0)}\left[2T_0\left(e^{j4\pi f_0 T_0} - e^{-j4\pi f_0 T_0}\right) + \frac{1}{j2\pi f_0} + \frac{1}{j2\pi f_0} - \frac{1}{j2\pi f_0}\left[e^{j4\pi f_0 T_0} + e^{-j4\pi f_0 T_0}\right]\right]$$

$$=\frac{A^2}{\pi f_0}\sin 4\pi f_0 T_0+\frac{A^2}{jT_0\pi f_0}\left[2T_0\sin\left(4\pi f_0 T_0\right)\right]+\frac{A^2}{2T_0\left(-j2\pi f\right)}\left[\frac{2}{j2\pi f}-\frac{2\cot 4\pi fT_0}{j2\pi f}\right]$$

$$=\frac{2A^2}{2T_0}-\frac{2\sin^2 2\pi fT_0}{\left(j2\pi f\right)^2}$$

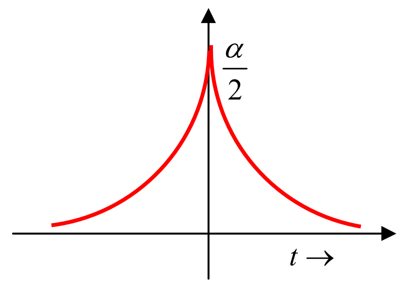
$$H\left(f\right)=2A^2T_0\frac{\sin^2 2\pi fT_0}{\left(2\pi fT_0\right)^2}=2A^2T_0\left(\frac{\sin 2\pi fT_0}{2\pi fT_0}\right)^2$$

**Example:** If $h\left(t\right)=\frac{1}{2}\alpha e^{-\alpha|t|}$

Such that $h\left(t\right)=\frac{1}{2}\alpha e^{\alpha t}$ $\quad t<0$

$=\frac{1}{2}\alpha e^{-\alpha t}$ $\quad t>0$

Then find $H\left(f\right)$




**Solution:** $F\left[h\left(t\right)\right]=H\left(f\right)=\int_{-\infty}^{\infty}\frac{1}{2}\alpha e^{-\alpha|t|}e^{-j2\pi ft}dt$

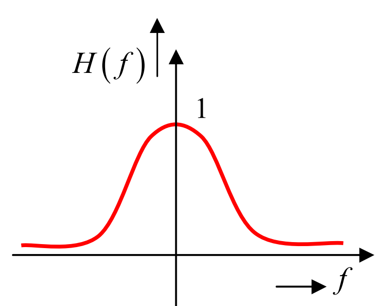
$$=\int_{-\infty}^{0}\frac{1}{2}\alpha e^{\alpha t}e^{-j2\pi ft}dt+\int_{0}^{\infty}\frac{1}{2}\alpha e^{-\alpha t}e^{-j2\pi ft}dt$$

$$=\frac{\alpha}{2}\left[\frac{e^{(\alpha-j2\pi f)t}}{\alpha-j2\pi f}\right]_{-\infty}^{0}+\frac{\alpha}{2}\left[\frac{e^{-(\alpha+j2\pi f)t}}{-\left(\alpha+j2\pi f\right)}\right]_{0}^{\infty}$$

$$H\left(f\right)=\frac{\alpha^2}{\alpha^2+4\pi^2 f^2}$$

**Or**

**Solution:** $H\left(f\right)=\int_{-\infty}^{\infty}\frac{1}{2}\alpha e^{-\alpha|t|}e^{-j2\pi ft}dt$




$$=\int_{-\infty}^{\infty}\frac{\alpha}{2}e^{-\alpha|t|}\left(\cos 2\pi ft-j\sin 2\pi ft\right)dt=0$$

$$=\int_{-\infty}^{\infty}\frac{\alpha}{2}e^{-\alpha|t|}\cos 2\pi ftdt$$

$$=\alpha\left\{\frac{e^{-\alpha t}}{\alpha^2+4\pi^2 f^2}\left[-\alpha\cos 2\pi ft+2\pi f\sin 2\pi ft\right]_0^{\infty}=\frac{\alpha}{\alpha^2+4\pi^2 f^2}\left[0-\left(-\alpha\right)\right]\right.$$

$H(f)=\dfrac{\alpha^2}{\alpha^2+4\pi^2 f^2}$ , Which is a Lorentzian function.

**Solution of a Differential equation using Fourier Transformation:**

Consider $\dfrac{d^2y}{dx^2}-\omega^2 y=-\delta(x-a)$

Fourier transformation of a derivative, if $h(t)$ and $h(f)$ forms Fourier transformation pair

Then $h(t)\Leftrightarrow H(f)$

(a) $\dfrac{d^n h(t)}{dt^n}\Leftrightarrow (j2\pi f)^n H(f)$

(b) $(-j2\pi ft)^n S(t)\Leftrightarrow \dfrac{d^n H(f)}{df^n}$

where $S(t)$ is something different from $h(t)$

Example: Find the Fourier transform of $\delta'(x)$.

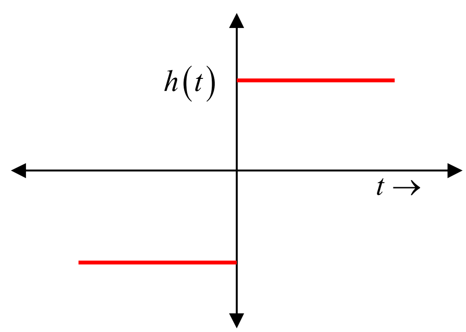
Solution: $F(k)=\int_{-\infty}^{\infty}\delta'(x)e^{ikx}dx$

$=0-ik=(-ik)^1\cdot 1=-ik$

$\Rightarrow f(x)\Rightarrow F(k)\Rightarrow \dfrac{d^n f(x)}{dx^n}=(-ik)^n F(k)$

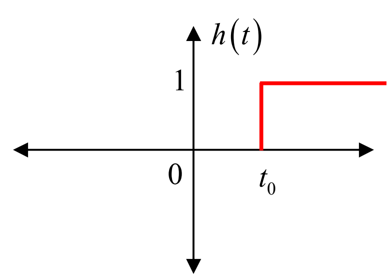
- **Sign function:** $\operatorname{sgn}(t)=\begin{cases}-1 & t<0\\ 1 & t>0\end{cases}$

  So $F\{\operatorname{sgn}(t)\}=\dfrac{2}{j\omega}$




- **Unit step function**

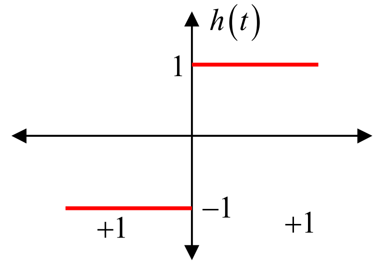
  $h(t)=u(t-t_0)$



We can we the linear property of Fourier Transform to get Fourier transform of other function.

If we add $1$ in $\operatorname{sgn}(t)$ then using the linear property we get:

$$\operatorname{sgn}(t)+1=2u(t)\Rightarrow u(t)=\frac{1}{2}\left(1+\operatorname{sgn}(t)\right)$$




$$\Rightarrow F\left[u(t)\right]=\frac{1}{2}\left(\delta(f)+\frac{2}{j\omega}\right)=\frac{1}{j\omega}+\frac{1}{2}\delta(f)$$

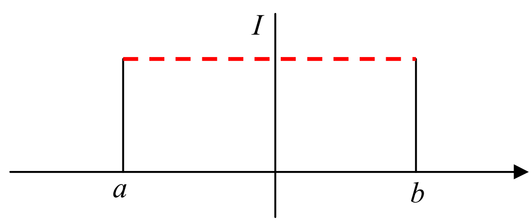
$$F\left[u(t)\right]=\frac{1}{j\omega}+\pi\delta(\omega)$$

$$F(1)=2\pi\delta(\omega)=\delta(f)$$

- **Barrier function in terms of unit step function:**

Barrier function can be written in terms of a unit step function.

i.e. $u(x-a)-u(x-b)$



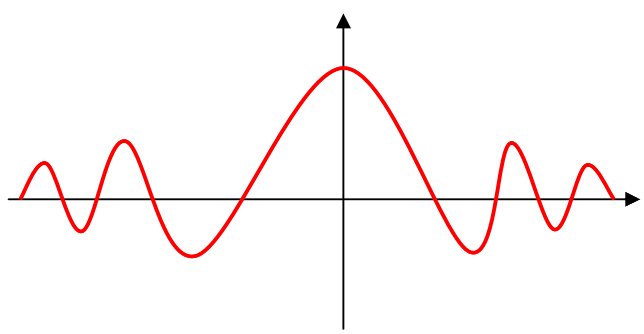

if $u(t)\Leftrightarrow\dfrac{1}{j\omega}+\pi\delta(\omega)$

Then $u(t-t_0)\Leftrightarrow e^{-j\omega t_0}\left(\dfrac{1}{j\omega}+\pi\delta(\omega)\right)$

$\Rightarrow$ We can find the Fourier transform of barrier function using the unit step function:

- **Sine function:**

$$f(x)=\frac{\sin bx}{x}$$



Then $F\{f(x)\}=?$

when $\tilde{f}(x)=\int_{-\infty}^{\infty}dx\,f(x)e^{ikx}$

$$=\int_{-\infty}^{\infty}dx\left(\frac{\cos kx\sin bx}{x}+ik\sin kx\frac{\sin bx}{x}\right)dx$$

$$=\int_{-\infty}^{\infty}\left(\frac{-\sin(kx-bx)}{2x}+\frac{\sin(kx+bx)}{2x}\right)dx$$

$$=\int_{-\infty}^{\infty}\frac{-\sin(k-b)dx}{2x}+\int_{-\infty}^{\infty}\frac{\sin(k+b)dx}{2x}$$

$$\tilde{f}(x)=\pi\left[u(k+b)-u(k-b)\right]$$