Multiple Variable Calculus

4. Taylor Series

About a point x_0.

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}' f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots$$

Linear approximation:

$$\Rightarrow f(x) = f(x_0) + \frac{(x-x_0)'}{1!} f'(x_0)$$

Example: $f(x) = \sin x, x_0 = \pi/4 = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$\sin x = \frac{1}{\sqrt{2}}\left(1 + x - \pi/4\right)$$

Example: $f(x) = \sin x, x_0 = 0 = 0 + \left(x - 0\right)\cos 0$

$$\sin x = x$$

$$f\left(x\right) = \log\frac{e^x + e^{-x}}{2}\cosh x, x_0 = 0$$

$$f\left(x\right) = \cos$$

$$f\left(0\right) = 0$$

$$f'\left(x\right) = \frac{1}{\cosh x}\sinh x = \tanh x$$

$$f'\left(0\right) = 0$$

$$f''\left(x\right) = 1 - \tanh^2 x = 1 - \left[f'\left(x\right)\right]^2$$

$$f''\left(0\right) = 1 - 0$$

$$f''\left(0\right) = 1$$

$$f'''\left(x\right) = -2f'\left(x\right)f''\left(x\right)$$

$$f'''\left(0\right) = 0$$

$$f^{iv}\left(x\right) = -2\left[0 \times 0 + 1\right]$$

$$f^{iv}\left(x\right) = -2$$

$$f^{v}\left(x\right) = -2\left[f'\left(x\right)f^{iv}\left(x\right) + f'''\left(x\right)f''\left(x\right) + 2f''\left(x\right)f'''\left(x\right)\right]$$

$$= -2\left[0 + 0 + 0\right]$$

$$f^{v}\left(x\right) = 0$$

$$f^{vi}\left(x\right) - 2\left[f'\left(x\right)f^{v}\left(x\right) + f^{iv}\left(x\right)f''\left(x\right) + f'''\left(x\right)^2 + f''\left(x\right)f^{iv}\left(x\right)\right.$$

$$\left. + 2f''\left(x\right)f^{iv}\left(x\right) + 2f'''\left(x\right)f'''\left(x\right)\right]$$

$$= -2\left[0 + \left(-2\right) + 0\left(-2\right) - 4 + 0\right] = \left(-8\right) \times \left(-2\right)$$

$$f^{vi}\left(x\right) = 16$$

$$f(x)=f(x_0)+\frac{(x-x_0)'}{11}f'(x_0)+\frac{(x-x_0)}{2!}f''(x_0)+\frac{(x-x_0)}{3!}f'''(x_0)+\cdots$$

$$=0+0+\frac{(x-0)^2}{2!}(1)+0+\frac{(x-0)^4}{4!}(-2)+0+\frac{(x-0)^6}{6!}(16)$$

$$=\frac{x^2}{2!}-2\frac{x^4}{4!}+16\frac{x^6}{6!}-\cdots\cdots$$

Example: Given $f(1)=1$

$f'(1)=1$

$f''(1)=1$

$f(1/2)=?$

Taylor Series

Let $x_0=1$

$$f(x)=f(x_0)+\frac{(x-1)}{1!}f'(1)+\frac{(x-1)^2}{2!}f''(1)+\cdots\cdot$$

$$f(x)=f(1)+\frac{(x-1)}{1!}f'(1)+\frac{(x+1)^2}{2}f''(1)$$

$\therefore f(1/2)$

$$=1+-1/2+1/8=\frac{8+4+1}{8}=\frac{5}{8}$$

$$f(1/2)=\frac{5}{8}=0.625$$

Example: $f(x)=\sin(\sin x), x_0=0$

$f(0)=0$

$f'(x)=\cos(\sin x)\cos x$

$f'(0)=1$

$f''(x)=\cos(\sin x)(-\sin x)+\cos x-\sin(\sin x)\cos x$

$=(-\sin x)\cos(\sin x)+(-\cos^2 x)\sin(\sin x)$

$f''(0) = 0$

$f'''(x) = (-\sin x) - \sin(\sin x)(\cos x) + \cos(\sin x)(-\cos x)$

$+(-\cos^2 x)\cos(\sin x)(\cos x) + \sin(\sin x)(+2\cos x)(\sin x)$

$f'''(0) = -2$

Coefficients of $x^3 = \frac{-2}{3!} = \frac{-2}{6} = \frac{-1}{3}$

$f(x) = 0 + \frac{x}{1!}(1) + 0 + \frac{x^3}{3!}(-2)$

$f(x)$ One Variable (x_0)

$f(x, y)$ two variable expansions about (x_0, y_0)