Matrices

4.9 Pauli Spin Matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) $\sigma_j \sigma_k = i\sigma_l$

$$(j,k,l) \rightarrow (1,2,3),(2,3,1),(3,1,2)$$

$$\Rightarrow \sigma_1\sigma_2 = i\sigma_3$$

(a) $(\sigma_i)^2 = ?$

$$(\sigma_1)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\sigma_2)^2 = \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$(\sigma_3)^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

(b) $|\sigma_i| = -1$,Orthogonal Matrix

(c) $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$, $\delta_{ij} = 1$ if $i = j$

$= 0$ if $i \neq j$

$\delta_{ij} =$ Kronecker Delta, $\hat{i} \neq j$

$\sigma_1\sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_2\sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$\Rightarrow \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$

For $i = j \Rightarrow \sigma_1^2 + \sigma_2^2 = 2I$

Question: If P & Q is Real Symmetric, then which of the following option is correct for PQ?

(a) Symmetric for all P and Q

(b) Never Symmetric

(c) Symmetric if $PQ = QP$

(d) Anti Symmetric for all P and Q

Answer: (c)

Solution: $(PQ)^T = Q^T P^T = QP$

Question: If $M = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$, the eigenvalues of M are?

(a) Real and positive

(b) Purely imaginary with mod 1

(c) Complex with mod 1

(d) Real and negative

Answer: (c)

Solution: $|M - \lambda I| = 0 \quad \Rightarrow \begin{bmatrix} \frac{i}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} - \lambda \end{bmatrix} = 0$

$\left(\frac{i}{\sqrt{2}}-\lambda\right)^2-\left(\frac{1}{\sqrt{2}}\right)^2=0 \qquad \Rightarrow\left(\frac{\hat{i}}{\sqrt{2}}-\lambda-\frac{1}{\sqrt{2}}\right)\left(\frac{\hat{i}}{\sqrt{2}}-\lambda+\frac{1}{\sqrt{2}}\right)=0$

$\lambda_1=\frac{i+1}{\sqrt{2}}, \qquad \lambda_2=\frac{i-1}{\sqrt{2}}$

$\Rightarrow i^{\dagger}=-i$ (Conjugate)

Question: Given two $(n\times n)$ matrices such that $\hat{P}$ is Hermitian and $\hat{Q}$ is skew Hermitian Matrix.

Then which of the following combination is a Hermitian matrix.

(a) $\hat{P}\hat{Q}$ (b) $i\hat{P}\hat{Q}$ (c) $\hat{P}+i\hat{Q}$ (d) $\hat{P}+\hat{Q}$

Answer: (c)

Solution: (a) $\left(\hat{P}\hat{Q}\right)^{\dagger}=Q^{\dagger}P^{\dagger}=-QP$

(b) $\left(i\hat{P}\hat{Q}\right)^{\dagger}=i^{\dagger}Q^{\dagger}P^{\dagger}=(-i)(-Q)(P)=iQP$

(c) $\left(\hat{P}+i\hat{Q}\right)^{\dagger}=P^{\dagger}+i^{\dagger}Q^{\dagger}=P+(-i)(-Q)=\hat{P}+i\hat{Q}$

(d) $\left(\hat{P}+\hat{Q}\right)=P^{\dagger}+Q^{\dagger}=P-Q$

Question: For the three matrices given below, which one of the options is correct?

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

(a) $\sigma_1\sigma_2=-i\sigma_3$

(b) $\sigma_1\sigma_2=i\sigma_3$

(c) $\sigma_1\sigma_2+\sigma_2\sigma_1=I$

(d) $\sigma_3\sigma_2=-i\sigma_1$

Answer: (b)

Solution: $\sigma_1\sigma_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}=i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$\sigma_1\sigma_2=i\sigma_3$

Question: If $M=\begin{bmatrix}4 & x\\ 6 & 9\end{bmatrix}$ and $\det M=0$. Then which of the following option is correct for Matrix M?

(a) M is symmetric

(b) M is invertible

(c) One eigenvalue is 13

(d) Its eigenvectors are orthogonal

Answer: (a)

Solution: $M=\begin{bmatrix}4 & x\\ 6 & 9\end{bmatrix}\Rightarrow \det M=\begin{vmatrix}4 & x\\ 6 & 9\end{vmatrix}=36-6x=0\Rightarrow x=6$

$M=\begin{bmatrix}4 & 6\\ 6 & 9\end{bmatrix}$

For eigenvalue:

$\begin{bmatrix}4-\lambda & 6\\ 6 & 9-\lambda\end{bmatrix}=0 \quad \Rightarrow (4-\lambda)(9-\lambda)-36=0$

$\Rightarrow \lambda^2-13\lambda=0 \;\; So \;\; \lambda_1=13,\lambda_2=0$

For eigenvector:

$A-13I=\begin{bmatrix}-9 & 6\\ 6 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$-9x_1+6x_2=0,\ 6x_1-4x_2=0\Rightarrow -3x_1+2x_2=0$

Let $x_1,=1$ then $x_2=\frac{3}{2}\Rightarrow X_1=\begin{bmatrix}1\\ 3/2\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$

Question: $M=\begin{bmatrix}3 & i & 0\\ -i & 3 & 0\\ 0 & 0 & 6\end{bmatrix}$, What will be its eigenvalue?

(a) $2,4,6$

(b) $2i,4i,6$

(c) $2i,4,8$

(d) $0,4,8$

Answer: (a)

Solution: $T_r(A)=3+3+6=12$

$|M|=3(18)-i(-6i)+0=54-6=48$

And determinant is the product of eigenvalues.