Multiple Variable Calculus

5. Taylor Series in 2D

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$+\frac{f_{xx}(x_0,y_0)}{2!}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)+\cdots$$

Taylor series nth degree polynomial

$$P_n(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(a,b)\cdot\frac{(x+a)^i (y-b)^j}{i!\,j!}$$

$n = 2$

$$P_2(x,y) = \sum_{i=0}^{2}\sum_{j=0}^{2-i}\frac{\partial^{i+j}}{\partial x^i \partial y^j}\left[f(x_0,y_0)\right](x-x_0)^i (y-y_0)^j$$

$\Rightarrow$ I will start at 0 and continue to increase upto 2

$\Rightarrow$ I will start at 0 and continue to increase to $2-i$

$$\frac{\partial}{\partial x\partial y}=\frac{\partial}{\partial y\partial x} \quad f_y=\frac{\partial}{\partial y}$$

$H=2$

$i=0, j=0 \qquad i=1, j=0 \qquad i=2, j=0$

$i=0, j=1 \qquad i=1, j=1$

$i=0, j=2$

$$P_2(x,y)=\frac{f(x_0,y_0)}{0!0!}(x-x_0)^0(y-y_0)^1+\frac{f(x_0,y_0)}{0!1!}(x-x_0)^0(y-y_0)^1$$

$$+\frac{f_{yy}(x_0,y_0)}{0!2!}(x-x_0)^1(y-y_0)^2+\frac{f_x(x_0,y_0)}{1!0!}(x-x_0)^1(y-y_0)^2$$

$$+\frac{f_{xy}(x_0,y_0)}{1!1!}(x-x_0)^1(y-y_0)^2+\frac{f_{xx}(x_0,y_0)}{2!0!}(x-x_0)^2(y-y_0)^0$$

$$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)$$

$$+f_y(x_0,y_0)(y-y_0)+f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$

$$+\frac{f_{xx}(x_0,y_0)}{2!}(x-x_0)^2+\frac{f_{yy}(x_0,y_0)}{2!}(y-y_0)$$

$\left[f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)\right]$ Linear Approx.

Example: $f(x,y)=x\sqrt{y}, P(1,4)=?$

Write linear approximation for $f(x,y)$ using Taylor's Series

$$f_x(x,y)=\frac{\partial}{\partial x}\left(x\sqrt{y}\right)=\sqrt{y}$$

$$f_x(1,4)=\sqrt{4}=2$$

$$f_y(x,y)=\frac{\partial}{\partial y}\left(x\sqrt{y}\right)=\frac{1}{2}\frac{x}{y}$$

$$f_y(1,4)=\frac{1}{4}=\frac{1}{2}\cdot\frac{1}{\sqrt{4}}=\frac{1}{4}$$

$$f(x_0,y_0)=1\sqrt{4}=2$$

$$f(x,y)=2+2(x-1)+\frac{1}{4}(y-4)$$

$$f(x,y) = 2x + \frac{y}{4} - 1$$

Example: $f(x,y) = e^x \cos y \qquad P(\ \ ) = ?$

$$f(0,0) = 1$$

$$f(x,y) = f(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

$$\tan^{-1} 1 = x$$

$$f_x = e^x \cos y$$

$$f_x = (0,0) = 1$$

$$f_y = -e^x \sin y \quad f_y(0,0) = 0$$

$$f_{xx} = e^x \cos y \quad (0,0) = 1$$

$$f_{yy} = -e^x \cos y = -1 = 1 + (x - 0) + 0(y - 0)$$

$$f(x,y) = 1 + x$$

Example: $f(x,y) = \tan^{-1}(x + 2y) \qquad P(1,0) = ?$

$$f(1,0) = \frac{\pi}{4}$$

$$L(x,y) = f(1,0) + f_x(x - x_0) + f_y(y - y_0)$$

$$f_x = \frac{\partial}{\partial x}\tan^{-1}(x + 2y) = \frac{1}{1 + (x + 2y)^2}$$

$$f_x(1,0) = \frac{1}{2}, \quad f_y = \frac{1}{1 + (x + 2y)^2}(+2)$$

$$f_y(1,0) = \frac{+2}{2} = +1 = \frac{\pi}{4} + \frac{1}{2}(x - 1) + 1(y - 0)$$

$$L(x,y) = \frac{\pi}{4} + \frac{1}{2}(x - 1) + y$$

$$f_{xx} = \left(1 + 2(x + 2y)^2\right)^{-1} = -1\left(1 + (x + 2y)^2\right)^{-2} \cdot \left(0 + 2(x + 2y)\right)$$

$$f_{xx}(1,0) = \frac{-2(x + 2y)}{\left(1 + (x + 2)^2\right)^2} = \frac{-2(1)}{(1 + 1 + 0)^2} = \frac{-2}{4} = \frac{-1}{2}$$

$$f_{yy}=\frac{-4(x+2y)}{\left(1+(x+2y)^2\right)^2}=\frac{-4(1)}{(1+1)^2}=-1$$

$$f_{xy}=\frac{\partial}{\partial x\partial y}\tan^{-1}(x+2y)=\frac{\partial}{\partial x}\left(\frac{+2}{1+(x+2y)^2}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{+2}{1+(x+2y)^2}\right)=2\frac{\partial}{\partial x}\left(1+(x+2y)^2\right)^{-1}$$

$$\frac{\pi}{4}-\frac{3}{4}+x+2y-\frac{x^2}{4}-xy-y^2=2\frac{(-1)}{\left(1+(x+2y)^2\right)^2}\cdot 2(x+2y)$$

$$f_{xy}(1,0)=\frac{-2}{(1+1)^2}2(1)=\frac{-4}{4}-1$$

$$f(x,y)=f(x_0,y_0)+\frac{f_x(x,y)}{1!}(x-x_0)+\frac{f_y(x,y)}{1!}(y-y_0)+f_{xy}(x,y)(x-x_0)(y-y_0)$$

$$+\frac{f_{xx}}{2}(x,y)(x-x_0)^2+\frac{f_{yy}}{2}(x,y)(y-y_0)^2$$

$$=\frac{\pi}{4}+\frac{1}{2}(x-1)+y-(x-1)(y)+(-1)\left(\frac{1}{2}\right)y^2$$

$$=\frac{\pi}{4}+\frac{1}{2}x-\frac{1}{2}+y-xy+y-\frac{1}{4}x^2-\frac{1}{4}+\frac{1}{2}x-\frac{1}{2}y^2=\frac{\pi}{4}-\frac{3}{4}+x+2y-\frac{x^2}{4}-xy-y^2$$