Complex Analysis

5.7 Laurent's Series:

$$f(z)=\underbrace{a_0+a_1(z-z_0)+a_2(z-z_0)}_{Analytic\ part}+\cdots\cdots+\underbrace{\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\frac{b_3}{(z-z_0)^3}}_{principle\ part}+\cdots$$

For Laurent's series, z_0 is a singular point.

If $b_n \to 0$ for all n=1,2,3…….., Then Laurent's series reduces to Taylor's series.

Singularity:

1. z_0 is essential singularity if $b_1, b_2, b_3, \ldots, b_n, \ldots\ldots\infty$ are infinite in number.

2. z_0 is called non-essential singularity if $b_1, b_2, b_3, \ldots$ are finite in number.

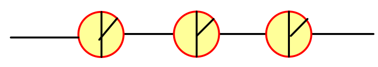
3. If only b_1 is there at $z=z_0$ then z_0 is a simple pole.

4. The point z_0 at which the function $f(z)$ is not defined, infinite or multi-valued then the point z_0 is known as singular point.

5. If only b_n is there at $z = z_0$, that is known as n^{th} order pole.

6. **Isolated Singularity**: When for a function $f(z)$ there are two or more singularities then in the neighborhood of one singularity no other singularity will exist and that is known as Isolated Singularity.

Example: $f(z) = \dfrac{1}{\sin z}$, $\sin z = \sin n\pi$,



$z = n\pi$, $z = 0, \pi, -\pi, 2\pi, -2\pi \cdots\cdots$

7. **Non-Isolated Singularity:** When for a function $f(z)$ there are two or more singularities then in the neighborhood of one singularity, other singularities can exist and that is known as non-isolated Singularity.

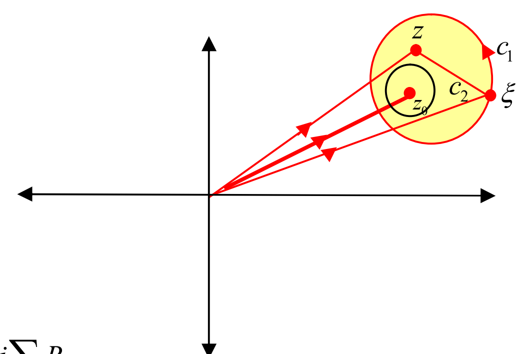
Example: $f(z) = \dfrac{1}{\sin\dfrac{\pi}{z}}$, $\dfrac{\pi}{z} = n\pi$, $z = \dfrac{1}{n}$, $z = 1, \dfrac{1}{2}, \dfrac{1}{3}, \cdots\cdots$

8. Branch point singularity: If $f(z) = (z - z_0)^{\frac{1}{2}}$, then z_0 is a branch point singularity. The function $f(z)$ is multi-valued at $z = z_0$ and this is the identity crisis.

Example: $f(z) = \log_e z$ & $f(z) = z^{1/2}$ so $z_0 = 0$ is a branch point.

$\Rightarrow$ **Laurent's Series:** $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n'=1}^{\infty} \dfrac{b_{n'}}{(z - z_0)^{n'}}$

$$a_n = \frac{1}{2\pi i} \oint_{c_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$



$$b_n = \frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)}{(\xi - z_0)^{-n'+1}} d\xi$$

$$b_1 = \frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)}{(\xi - z_0)^0} d\xi = \frac{1}{2\pi i} \int_{c_2} f(\xi) d\xi$$

$$\oint_c f(\xi) d\xi = 2\pi i b_1 \quad \Rightarrow \oint_c f(z) dz = 2\pi i b_1 = 2\pi i \sum R$$

$\Rightarrow$ The above equation is Cauchy's Integral Formula, where b_1 is Residue.

b_1 is the coefficients of $\dfrac{1}{z - z_0}$.

Example: Find the Laurent series for the following functions

(a) $f(z)=\dfrac{1}{z(z-1)}, \qquad |z|<1$

Solution: $f(z)=\dfrac{-1}{z}(1-z)^{-1}=\dfrac{-1}{z}(1+z+z^2+z^3+......)$

$$f(z)=-\frac{1}{z}-1-z-z^2\cdots\cdots$$

$$f(z)=\underbrace{-1-z-z^2}_{Analytic\ part}\cdots\cdots-\underbrace{\frac{1}{z}}_{principle\ part}, \qquad b_1=-1$$

- So it is non-essential
- Isolated
- Pole of order 1 i.e. simple pole.

(b) $f(z)=\dfrac{1}{z(z-1)}, \ |z|>1$

Solution: $f(z)=\dfrac{1}{z(z-1)}, \ |z|>1 \quad \Rightarrow \dfrac{1}{|z|}<1$

$$\frac{1}{z^2\left(1-\frac{1}{z}\right)}=\frac{1}{z^1}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$f(z)=\dfrac{1}{z^2}+\dfrac{1}{z^3}+\dfrac{1}{z^4}+\dfrac{1}{z^5}+\cdots\cdots$ Principle part and Analytic parts $=0$, $b_1=0$

- So it is an essential singularity.
- Pole of order 2 i.e. second-order pole.

(c) $f(z)=\dfrac{1}{z(z-1)}, \ |z|>1$ at $z_0=1$

Solution: $z_0=1, \ \ z-1=t, \qquad z=1+t$

$$f(z)=\frac{1}{t(1+t)}=\frac{1}{t}(1+t)^{-1}=\frac{1}{t}\left(1-t+t^2-t^3+\cdots\right)=\frac{1}{t}-1+t-t^2+\cdots\cdots$$

$$=\frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots\cdots$$

(d) $f(z)=\frac{1}{z(z-2)}$

(a) $0<|z|<2, \quad \frac{|z|}{2}<1$

Solution: $f(z)=\frac{1}{z(z-2)}$

$$f(z)=\frac{-1}{2z}\left[\frac{1}{1-\frac{z}{2}}\right]=-\frac{1}{2z}\left[1-\frac{z}{2}\right]^{-1}=-\frac{1}{2z}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\cdots\right]$$

So $b_1=-\frac{1}{2}$ i.e. non-essential singularity.

(b) $2<|z|<\infty, \quad \frac{2}{|z|}<1$

Solution: $f(z)=\frac{1}{z(z-2)}$

$$f(z)=\frac{1}{z(z-2)}=\frac{1}{z^2\left(1-\frac{2}{z}\right)}=\frac{1}{z^2}\left[1-\frac{2}{z}\right]^{-1}$$

$$f(z)=\frac{1}{z^2}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\cdot\right]=\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\frac{8}{z^5}+\cdots\cdot, \quad b_1=0$$

- essential singularity.
- Analytic part $=0$

Example: Find the residue of the following function $f(z)=\frac{1}{(z-1)(z-2)}, \quad 1<|z|<2 \quad z_0=1$.

Solution: $f(z)=\frac{1}{(z-1)(z-2)}, \quad 1<|z|<2$ so $\frac{1}{|z|}<1, \ \frac{|z|}{2}<1$

$$(z-z_0)=(z-1)$$

$$f(z)=\frac{1}{t(t-1)}=\frac{-1}{t}(1-t)^{-1} \quad \begin{bmatrix} 0<|z-1|<1 \\ 0<|t|<1 \end{bmatrix}$$

$$=-\frac{1}{t}\left[1+t+t^2+t^3+\cdots\cdots\right]=-\frac{1}{t}-1-t-t^2-t^3\cdots\cdot$$

$=-\frac{1}{z-1}-1-(z-1)-(z-1)^2-(z-1)^3\cdots\cdots,\ b_1=-1$, non-essential singularity.

Example: Find the order and singularity for the function $f(z)=\frac{1}{z^5}\sin z$ at $z_0=0$.

Solution: $f(z)=\frac{1}{z^5}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}\cdots\right]=\frac{1}{z^4}-\frac{1}{z^2 3!}+\frac{1}{5!}-\frac{z^2}{7!}-\frac{z^4}{9!}\cdots\cdots$

So residue, $b_1=0$ and order are 4.

Example: Find the residue of the function $f(z)=z^2e^{1/z}$ at $z_0=0$.

Solution: $f(z)=z^2\left[1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots\cdots\right]=z^2+z+\frac{1}{2!}+\frac{1}{3!z}+\cdots\cdot$

So residue $b_1=\frac{1}{6}$

Example: Find the order and residue for the function $f(z)=\frac{1}{z^3-z^4}$ at $z_0=0$.

(a) $0<|z|<1$

Solution: $0<|z|<1 \quad \Rightarrow z<1$

$$f(z)=\frac{1}{z^3(1-z)}=\frac{1}{z^3}(1-z)^{-1}=\frac{1}{z^3}\left[1+z+z^2+z^3+z^4+\ldots.\right]$$

$$f(z)=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\cdots\cdots$$

So residue $b_1=1$ & Third order

(b) $|z|>1$

Solution: $|z|>1 \ \Rightarrow \frac{1}{|z|}<1$

$$f(z)=\frac{1}{-z^4\left(1-\frac{1}{z}\right)}=-\frac{1}{z^4}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{z^4}-\frac{1}{z^5}-\frac{1}{z^6}-\frac{1}{z^7}\cdots\cdots$$

So residue, $b_1=0$

Example: $f(z)=\dfrac{-2z+3}{z^2-3z+2}$

Solution: $f(z)=\dfrac{-2z+3}{z^2-3z+2}=\dfrac{-2z+3}{(z-1)(z-2)}=\dfrac{A}{(z-1)}+\dfrac{B}{(z-2)}$

$-2z+3 = Az-2A+Bz-B = z(A+B)-2A-B$

On comparing coefficients on both sides we get A=-1 & B=-1

$$f(z)=\frac{-1}{z-1}-\frac{1}{z-2}=f_1(z)+f_2(z)$$

Here we have two functions and we will see that different functions will give different series in a different region.

$$f_1(z)=-\frac{1}{z-1},\quad f_2(z)=-\frac{1}{z-2}$$

(I) $f_{1a}(z)=\dfrac{1}{1-z} \qquad \Rightarrow |z|<1$

$$f_{1a}(z)=(1-z)^{-1}=1+z+z^2+z^3+z^4+\cdots \qquad (1)$$

$f_{1b}(z)=\dfrac{1}{1-z} \qquad \Rightarrow |z|>1$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$f(z)=-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}\cdots \qquad (2)$$

(2) $f_2(z)=-\dfrac{1}{z-2} \qquad \Rightarrow |z|<2$

$$\Rightarrow f_{2a}(z)=\frac{1}{(2-z)}=\frac{1}{2\left(1-\frac{z}{2}\right)}=\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}=\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\cdots\right)$$

$$f_{2a}(z)=\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\frac{z^3}{16}+\cdots$$

$\Rightarrow f_{2b}(z)=-\dfrac{1}{z-2} \qquad \Rightarrow |z|>2$

$$f_{2b}(z)=\frac{1}{-z\left(1-\frac{2}{z}\right)}=-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}=-\frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right] f_{2b}(z)=-\frac{1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\frac{8}{z^4}\cdots\cdots$$

$$\Rightarrow f(z)=f_1(z)+f_2(z)$$

(1) $|z|<1, \quad f(z)=f_{1a}(z)+f_{2a}(z)$

(2) $1<|z|<2, \quad f(z)=f_{1b}(z)+f_{2a}(z)$

(3) $|z|>2, \quad f(z)=f_{1b}(z)+f_{2b}(z)$

Example: Find the Laurent series or Taylor series for $f(z)=\frac{1}{(z-1)(z-2)}$ in the following region:

(a) $|z|<1$

Solution: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}$

$$1=Az-2A+Bz-B, \quad A=-1, B=1$$

$$f(z)=\frac{-1}{z-1}+\frac{1}{z-2}=1[1-z]^{-1}-\frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$f(z)=\left(1+z+z^2+z^3+\cdots\cdots\right)-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

$$f(z)=\left(1-\frac{1}{2}\right)+\left(z-\frac{z}{4}\right)+\left(z^2-\frac{z^2}{8}\right)+\cdots=\frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\cdots$$

(b) $|z|>2$

Solution: $f(z)=\frac{1}{(z-1)(z-2)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}$

$$f(z)=-\frac{1}{z-1}+\frac{1}{z-2}, \qquad \frac{2}{|z|}<1, \ \frac{1}{|z|}<1$$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{z\left(1-\frac{2}{z}\right)}=-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}+\frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$f(z)=-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right]$$

$$f(z)=-\frac{1}{z}+\frac{1}{z}-\frac{1}{z^2}+\frac{2}{z^2}-\frac{1}{z^3}+\frac{4}{z^3}\cdots\cdots=\frac{1}{z^2}+\frac{3}{z^3}+\cdots$$

(c) $0<|z-1|<1$

Solution: $f(z)=\dfrac{1}{(z-1)(z-2)}=\dfrac{1}{(z-1)(z-1-1)}$

$$f(z)=-\frac{1}{(z-1)}\left[1-(z-1)\right]^{-1}=\frac{-1}{z-1}\left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$$

$$f(z)=-\frac{1}{z-1}-1-(z-1)-(z-1)^2\cdots\cdots$$

Calculation of Residue at pole z_0:

There are a few methods to calculate residue:

1. b_1 in Laurent series is the residue of $f(z)$ at some pole z_0.
2. Residue for n^{th} order pole $\left[\dfrac{1}{(z-z_0)^n}\right]$: $\lim\limits_{z\to z_0}\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$

   And for simple pole (n=1) : $\lim\limits_{z\to z_0}(z-z_0)f(z)$
3. Suppose $f(z)=\dfrac{\phi(z)}{\psi(z)}$, ($\phi(z)$ and $\psi(z)$ are transcendental function)

   Check for conditions at pole z_0: $\psi'(z_0)\neq 0$ and $\phi(z_0)\neq 0$

   If the above conditions will satisfy then residue will be $=\lim\limits_{z\to z_0}\dfrac{\phi(z_0)}{\psi'(z_0)}$

   Question: Find the residue of the following functions :

(a) $f(z)=\dfrac{1}{(z-1)}$

Solution: $f(z)=\dfrac{1}{(z-1)}=\lim\limits_{z\to 1}(z-1)\times\dfrac{1}{(z-1)}=1$

This is the Laurent series where all other coefficients are zero so residue (b_1) =1.

(b) $f(z)=\dfrac{z^2}{z^2+a^2}$?

Solution: $f(z)=\frac{z^2}{z^2+a^2}=\frac{z^2}{(z+ia)(z-ia)}$, $z=\pm ia$, two poles are there and both are simple poles.

at $z=ia$

$$f(z)=\lim_{z\to ia}\frac{(z-ia)z^2}{(z+ia)(z-ia)}=\frac{z^2}{2ia}=\frac{(ia)^2}{2ia}=\frac{-a}{2i}$$, Thus Residue $b_1=\frac{ai}{2}$.

at $z=-ia$

$$f(z)=\lim_{z\to -ia}\frac{(z+ia)z^2}{(z-ia)(z+ia)}=\frac{(-ia)^2}{-2ia}=\frac{a}{2i}=\frac{-a}{2i}$$, Thus Residue $b_2=\frac{-ai}{2}$.

(c) $f(z)=\frac{\tan z}{z^2-1}$.

Solution: $f(z)=\frac{\tan z}{z^2-1}=\frac{\sin z}{\cos z(z-1)(z+1)}$, $z=1,-1,\frac{\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\frac{-3\pi}{2},\ldots\ldots$

at $z=1$: $f(z)=\lim_{z\to 1}\frac{(z-1)\tan z}{(z-1)(z+1)}=\frac{\tan 1}{2}$

at $z=-1$: $f(z)=\lim_{z\to -1}\frac{(z+1)\tan z}{(z+1)(z-1)}=\frac{\tan 1}{2}$

at $z=\frac{\pi}{2}$: $f(z)=\lim_{z\to\frac{\pi}{2}}\frac{\sin z}{\cos z(z^2-1)}$, $\phi(z)=\sin z$ & $\psi(z)=\cos z(z^2-1)$

$\phi\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1\neq 0$ &

$$\psi'(z)=-\sin z(z^2-1)+2z\cos z\Rightarrow\psi'\left(\frac{\pi}{2}\right)=1-\frac{\pi^2}{4}\neq 0$$

Residue $=\frac{\sin\frac{\pi}{2}}{1-\frac{\pi^2}{4}}=\frac{1}{1-\frac{\pi^2}{4}}=\frac{4}{4-\pi^2}$

(d) $f(z)=\frac{1}{(z^2+a^2)^2}$.

Solution: $f(z)=\frac{1}{(z^2+a^2)^2}=\frac{1}{(z+ia)^2(z-ia)^2}$

At $z = ia$

$$f(z) = \lim_{z \to ia} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z-ia)^2 \frac{1}{(z+ia)^2 (z-ia)^2}$$

$f(z) = \frac{d}{dz} \frac{1}{(z+ia)^2} = \frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{1}{4ia^3}$, So residue $b_1 = \frac{1}{4ia^3}$

At $z = -ia$

$$f(z) = \lim_{z \to -ia} \cdot \frac{1}{1!} \frac{d}{dz} (z+ia)^2 \frac{1}{(z-ia)^2 (z+ia)^2} = \frac{-2}{(z-ia)^3}$$

$f(z) = \frac{-2}{(-2ia)^3} = \frac{-2}{+8ia^3} = \frac{1}{4ia^3}$, So, residue $b_1 = -\frac{1}{4ia^3}$.