Complex Analysis

5.8 Complex Integration or Contour Integration

$\Rightarrow \oint_c f(z)dz = 2\pi i \sum R$: Anticlockwise

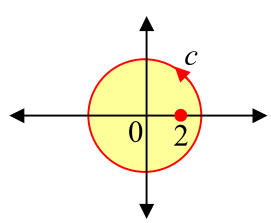
$\Rightarrow \oint_c f(z)dz = -2\pi i \sum R$: Clockwise

Example: Find the residue of the following functions:

(a) $f(z) = \oint_c \frac{2z-1}{z^2-z} dz$

Solution: $f(z) = \oint_c \frac{2z-1}{z^2-z} dz \quad \Rightarrow z^2 - z = z(z-1) = 0 \quad \Rightarrow z = 0,1$ are the poles

Now at $z = 0$, $f(z) = \lim_{z\to 0} \frac{(2z-1)\cdot z}{z(z-1)} = \frac{-1}{-1} = 1$

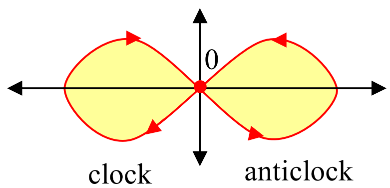


And at $z = 1$, $f(z) = \lim_{z\to 1} \frac{(z-1)(2z-1)}{z(z-1)} = \frac{1}{1} = 1$

Then Residue will be $2\pi i[1+1] \Rightarrow b = 4\pi i$.

(b) $f(z) = \oint_c \frac{dz}{z^2-1}$.

Solution: $(z-1)(z+1)=0, \quad z=1,-1$



At $z=1$: $f(z) = \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{z+1} = \frac{1}{2}$

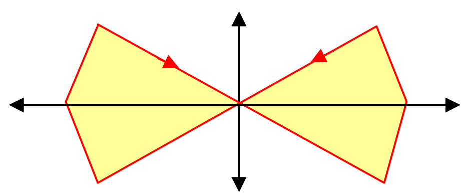
$\Rightarrow 2\pi i \times \frac{1}{2} = \pi i$ (Anticlockwise)

And at $z=-1$: $f(z) = \frac{(z+1)}{(z+1)(z-1)} = -\frac{1}{2}$

$\Rightarrow -2\pi i \times \left(-\frac{1}{2}\right) = \pi i$ (clockwise)

Residue $= \pi i + \pi i = 2\pi i$

(c) $f(z) = \oint_c \frac{ze^{i\pi z/2}}{z^2-1} dz$



Solution: $f(z) = \oint_c \frac{ze^{i\pi z/2}}{z^2-1} dz \quad \Rightarrow (z-1)(z+1)=0$

$z=1,-1$

At $z=1 \Rightarrow f(z) = \frac{(z-1)ze^{i\pi z/2}}{(z-1)(z+1)} = \frac{e^{i\pi/2}}{2} = \frac{i}{2} \Rightarrow \frac{i}{2}\times 2\pi i = -\pi$

At $z=-1 \Rightarrow f(z) = \frac{(z+1)ze^{i\pi z/2}}{(z-1)(z+1)} = (-1)\frac{e^{-i\pi/2}}{-2} = \frac{-i}{2} \Rightarrow \frac{-i}{2}\times -2\pi i = -\pi$

So residue will be R $= -\pi - \pi = -2\pi$.

(d) $f(z) = \int_c \frac{\cos z}{z} dz, \qquad |z|=1$

Solution: $f(z) = \int_c \frac{\cos z}{z} dz, \qquad z=0$, simple pole

at $z=0, \quad \cos 0^o = 1, \quad \Rightarrow I = 2\pi i \times 1 = 2\pi i$

(e) $f(z) = \int_c \frac{\sin z}{(z+4iz)} dz, \ \ c: |z-4-2i| = 6.5$

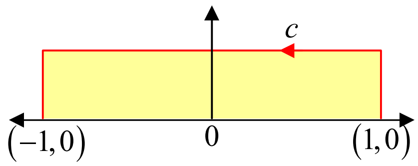
Solution: $f(z)=\int_c \frac{\sin z}{(z+4iz)}dz$, for pole $(z+4iz)=0 \Rightarrow z=0$

Residue at $z=0 \Rightarrow \lim_{z\to 0}\frac{z\sin z}{(z+4iz)}=0 \qquad \Rightarrow so\ I=2\pi i\times 0=0$

(f) $f(z)=\oint_c z^2e^z dz$

Solution: $f(z)=\oint_c z^2e^z dz=0$

$\Rightarrow \oint_c f(z)dz+\int_{-1}^{1} f(x)dx=0 \quad \Rightarrow \oint_c f(z)dz=-\int_{-1}^{1} f(x)dx$



$\oint_c f(z)dz=-\left[x^2e^x-2xe^x+2e^x\right]_{-1}^{1}=-e+\frac{5}{e}$

Residue $=\frac{5}{e}-e$

(g) $f(z)=\oint\frac{z^3dz}{z^2-5z+6}, \quad c:2|z|-5=0\Rightarrow |z|=\frac{5}{2}=2.5$

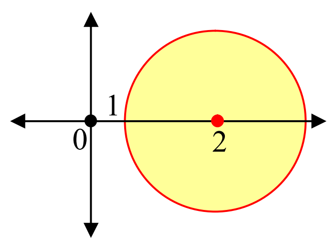
Solution: $z^2-5z+6=(z-2)(z-3),\ z=2,3$

z =2 is in the contour

$f(z)=\lim_{z\to 2}\frac{(z-2)z^3dz}{(z-2)(z-3)}=-8 \quad \Rightarrow \oint_c f(z)dz=-8\times 2\pi i=-16\pi i$

So residue will be $-16\pi i$.

(h) $f(z)=\oint\frac{e^z\sin z}{z^2}dz, \quad c:|z-2|=1$



Solution: $f(z)=\oint\frac{e^z\sin z}{z^2}dz$

$z=0$, pole which is outside of the contour, So Residue $=0$.

(i) $f(z)=\oint\frac{z^2}{e^z+1}dz,\ c:|z|=4$

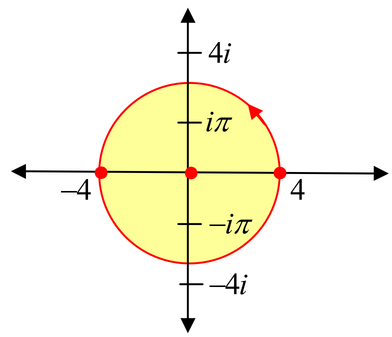
Solution: $f(z)=\oint\frac{z^2}{e^z+1}dz\Rightarrow e^z=-1=e^{i(2n+1)\pi}$

$z=i(2n+1)\pi \qquad \Rightarrow z=i\pi,3i\pi,5i\pi,7i\pi,\ldots$

$z_0 = i\pi$ (Inside)

$\phi(z) = z^2, \psi(z) = e^z + 1, \qquad \psi'(z) = e^z = e^{i\pi} = -1$

$f(z) = \dfrac{\phi(z)}{\psi'(z)} = \dfrac{(i\pi)^2}{-1} = \dfrac{-\pi^2}{-1} = \pi^2 \qquad \Rightarrow I_1 = 2\pi^3 i$



$z_0 = -i\pi$ (Inside)

$\phi(z) = z^2, \psi(z) = e^z + 1, \qquad \psi'(z) = e^z = e^{-i\pi} = -1$

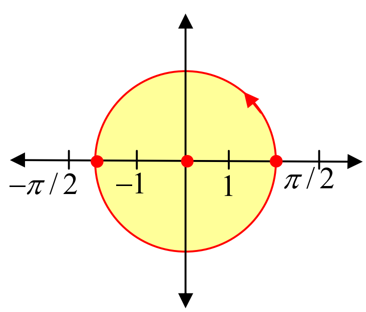
$f(z) = \dfrac{\phi(z)}{\psi'(z)} = \dfrac{(-i\pi)^2}{-1} = \dfrac{-\pi^2}{-1} = \pi^2 \qquad \Rightarrow I_2 = 2\pi^3 i$

$I = I_1 + I_2$, Thus Residue $I = 4\pi^3 i$

(j) $f(z) = \int_c \dfrac{\tan z}{z^2 - 1} dz, \qquad c : |z| = 1.5$

Solution: $f(z) = \int_c \dfrac{\tan z}{z^2 - 1} dz \Rightarrow z = \pm 1, \dfrac{\pi}{2}, -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \cdots$

$z = 1$ (Inside), $f(z) = \lim\limits_{z\to 1} \dfrac{\tan 1(z-1)}{(z-1)(z+1)} = \dfrac{\tan 1}{2}$



$z = -1$ (Inside), $f(z) = \lim\limits_{z\to -1} \dfrac{-\tan 1(z+1)}{(z-1)(z+1)} = \dfrac{\tan 1}{2}$

$\sum I = 2 \times \dfrac{\tan 1}{2} = \tan 1$

$\oint f(z) = 2\pi i \sum I = 2\pi i \tan(1)$

(k) $f(z) = \dfrac{1}{2\pi i} \oint_c \dfrac{e^{4z} - 1}{\cosh z - 2\sinh z} dz, \qquad |z| = 1$

Solution: $\psi(z) = \cosh z - 2\sinh z$ & $\phi(z) = e^{4z} - 1 \quad \Rightarrow \psi'(z) = \sinh z - 2\cosh z$

$\Rightarrow \cosh z - 2\sinh z = 0 \ \Rightarrow \cosh z = 2\sinh z \quad \Rightarrow \dfrac{e^z + e^{-z}}{2} = e^z - e^{-z}$

$e^z + e^{-z} - 2e^z + 2e^{-z} = 0 \quad \Rightarrow 3e^{-z} - e^z = 0 \Rightarrow 3e^{-z} = e^z \quad \Rightarrow e^{2z} = 3$

$f(z) = \dfrac{\phi(z)}{\psi'(z)} = \dfrac{e^{4z} - 1}{\sinh z - 2\cosh z} = \dfrac{(e^{2z})^2 - 1^2}{\sinh z - 2\cosh z}$

$$f(z)=\frac{8}{\left(\frac{e^z-e^{-z}}{2}\right)-\left(e^z+e^{-z}\right)}=\frac{8\times 2}{e^z-e^{-z}-2e^z-2e^{-z}}$$

$$f(z)=\frac{16}{-3e^{-z}-e^z}\Rightarrow\frac{-16}{\frac{3}{\sqrt{3}}+\sqrt{3}}=\frac{-16}{2\sqrt{3}}=-\frac{8}{\sqrt{3}}$$

$$f(z)=\frac{1}{2\pi i}\oint_c\frac{e^{4z}-1}{\cosh z-2\sinh z}dz=\ -\frac{8}{\sqrt{3}}$$

(I) $f(z)=\oint_c\frac{\log_e(z+1)}{z^2+1}\qquad c:|z-i|=1.4$

Solution: $f(z)=\frac{\log_e(z+1)}{z^2+1}=\frac{\log_e(z+1)}{(z+i)(z-i)}$

$z=i,-i$ are the singular points and $z=-1$ is branch point.

$$R=2\pi i\times\lim_{z\to i}\frac{(z-i)\log_e(z+1)}{(z+i)(z-i)}=2\pi i\times\frac{\log_e(i+1)}{2i}=\pi\log_e(i+1)$$

$$R=\pi\log_e\sqrt{2}e^{\frac{i\pi}{4}}$$