Multiple Variable Calculus

7. Other Relations

The relation between the pressure P temperature T and volume V of a certain amount of hydrogen gas may be expressed over a limited range by the equation $B \to B(T)$

$$P(V-B) = AT$$

(a) $\left(\frac{\partial V}{\partial T}\right)_P = \frac{A}{P} + \frac{\partial B}{\partial T}$ (b) $\left(\frac{\partial V}{\partial P}\right)_P = -\frac{A}{P^2}T$ (c) $dV(T,P)$

Solution: $V = B + \frac{AT}{P}$

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dt = \left(\frac{A}{P} + \frac{\partial B}{\partial T}\right)dT + \left(\frac{-A}{p^2}T\right)dP$$

- If $z = f(x,y)$

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1$$

This result is very peculiar for physics, especially Thermodynamics, where we can write the partial derivatives keeping some parameters constant into one or another form.

Proof: $z=f(x,y)$

$$dz=\left(\frac{\partial z}{\partial x}\right)_y dx+\left(\frac{\partial z}{\partial y}\right)_x dy \qquad (1)$$

$y=f(x,z)$

$$dy=\left(\frac{\partial y}{\partial x}\right)_z dx+\left(\frac{\partial y}{\partial z}\right)_x dz \qquad (2)$$

$x=f(y,z)$

$$dx=\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz$$

Substitute dy in (1)

$$dz=\left(\frac{\partial z}{\partial x}\right)_y dx+\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z dx+\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial z}\right)_x dz$$

Comparing Coefficients of dz ,

$$\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial z}\right)_x=1$$

Coefficients of dx

$$0=\left(\frac{\partial z}{\partial x}\right)_y+\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z-\left(\frac{\partial z}{\partial x}\right)_y=\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z\Rightarrow\left(\frac{\partial x}{\partial z}\right)_y\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_z=-1$$

- $f(x,y)=\sin^2(x+y)+x^2\cos y$
- $f(x,y,z)=x^3y^2z$

(i) $\dfrac{\partial^2}{\partial x\partial y}=?$ (ii) $\dfrac{\partial^2 f}{\partial y\partial x}=?$

(1) $\dfrac{\partial^2 f}{\partial x\partial y}$

$$\partial f = s\,\text{in}^2(x+y) + x^2\cos y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[2\sin(x+y) - x^2\sin y\cdot\cos(x+y)\right]$$

$$= 2\sin(x+y)\left(-\sin(x+y)\right) + 2\cos(x+y)\cos(x+y) - 2x\sin y$$

$$\frac{\partial^2 f}{\partial x\partial y} = 2\cos 2(x+y) - 2x\sin y$$

(b) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[2x^3yz\right] = 6x^2yz$

- $z = f(x^n y), n \neq 0$

$$x\frac{\partial z}{\partial x} = |n|, \qquad n \neq 0$$

$$x\frac{\partial z}{\partial x} = f'(x^n y)\cdot nx^n$$

$$y\frac{\partial z}{\partial y} = f'(x^n y)\cdot x^n$$

Read Basic Limit Problems of class 12 only elementary.