# Differential Equation

**8. Non-Homogeneous Linear ODEs of Second Order (RHS≠0)**

(i) DE with constant Coefficients

If the right-hand side in ODE is not 0, then the solutions can be found as follows: First, find the form of the solution of the corresponding homogeneous equation keeping the constants $C_1$ and $C_2$ as such: this is called the complementary solution $y_c(x)$;

Second, find a particular integral of the ODE $y_P(x)$.

Then the solutions of the ODE are of the form: $y(x) = y_c(x) + y_p(x)$. At this point only, you may determine the constants A and B from the boundary conditions.

There are two methods to find a particular integral of the ODE: the method of undetermined coefficients and the method of variation of parameters.

**Undetermined coefficients:**

This method consists in making an educated guess as to what the particular integral should look like. The following table can be used:

| $f(x)$ | **Particular Integral** |
|---|---|
| $k$ | $C$ |
| $kx$ | $Cx + D$ |
| $kx^2$ | $Cx^2 + Dx + E$ |
| $k\sin x$ or $k\cos x$ | $C\cos x + D\sin x$ |
| $k\sinh x$ or $k\cosh x$ | $C\cosh x + D\sinh x$ |
| $e^{kx}$ | $Ce^{kx}$ |
| $e^{rx}$, where r is a root of the characteristic equation | $Cxe^{kx}$ or $Cx^2e^{kx}$ |

The constants C and D are found by `plugging' the particular integral in the ODE, which will lead to conditions that define C and D.

**Example:**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2\sin 4x$$

**Solution:** We first find the complementary solution of the ODE. The characteristic equation is $r^2 - 5r + 6 = 0$ and the roots are $\frac{5\pm\sqrt{25-4\times6}}{2} = 3$ or 2. Therefore, the complementary solution is:

$$y_c(x) = Ae^{3x} + Be^{2x}$$

Then, we find a particular integral of the ODE. Since the right-hand side contains a $\sin 4x$, we look for a particular integral in the form $y_p(x) = C\cos 4x + D\sin 4x$. We want $y_p$ to be solution of the ODE so we must have:

$$\frac{d^2y_p}{dx^2} - 5\frac{dy_p}{dx} + 6y_p = 2\sin 4x$$

We have:

$$\frac{dy_p}{dx} = -4C\sin 4x + 4D\cos 4x$$
$$\frac{d^2y_p}{dx^2} = -16C\cos 4x - 16D\sin 4x$$

Putting back in the ODE:

$$(-16C\cos 4x - 16D\sin 4x) - 5(-4C\sin 4x + 4D\cos 4x) + 6(C\cos 4x + D\sin 4x) = 2\sin 4x$$

Re-arranging cos and sin:

$$(-16C - 20D + 6C)\cos 4x + (-16D + 20C + 6D)\sin 4x = 2\sin 4x$$
$$(-10C - 20D)\cos 4x + (-10D + 20C)\sin 4x = 2\sin 4x$$

The last equation must be true for any value of $x$, so we must have:

$$\begin{cases} -10C - 20D = 0 \\ 20C - 10D = 2 \end{cases}$$

$$\begin{cases} C = \dfrac{2}{25} \\ D = \dfrac{-1}{25} \end{cases}$$

So, a particular integral of the ODE is $y_p(x) = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x$ and the general solutions of the ODE are of the form: $y(x) = y_c(x) + y_p(x)$