Differential Equation

9. Linear Dependence and Independence of Solutions

Let the ODE $y'' + p(x)y' + q(x)y = 0$ have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of DE on I are linearly dependent on I if and only if their "Wronskian"

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

is 0 at some x_0 in I. Furthermore, if $W = 0$ at an $x = x_0$ in I, then $W = 0$ on I; hence, if there is an x_1 in I at which W is not 0, then y_1, y_2 are linearly independent on I.

Example $y'' + w^2 y = 0.$

Solution: $y_1 = \cos wx$ and $y_2 = \sin wx$ are solutions of DE. Their Wronskian is given by

$$w = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} = \omega\left(\cos^2\omega x + \sin^2\omega x\right) = \omega.$$

$w \neq 0$, means the solutions are linearly independent.

- For the $y'' + p(x)y' + q(x)y = 0$

Abel's formula $w = w(x_0)e^{-\int_{x_0}^{x} pdx}$

- $y_2 = y_1 \int \frac{w}{y_1^2} dx$

Abel's Formula

Let $y_1(x)$ and $y_2(x)$ be two solutions of

$$y'' + p(x)y' + q(x)y = 0$$

and let

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

be their Wronskian. We can compute the derivative $W'(x)$ as

$$\begin{aligned} W'(x) &= y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_1''(x)y_2(x) - y_1'(x)y_2'(x) \\ &= y_1(x)y_2''(x) - y_1''(x)y_2(x) \end{aligned}$$

Since y_1 and $y2$ are solutions, we have,

$$y_1''(x) = -p(x)y_1'(x) - q(x)y_1(x), \quad \text{and} \quad y_2''(x) = -p(x)y_2'(x) - q(x)y_2(x)$$

and substituting these, we get

$$\begin{aligned} W'(x) &= y_1(x)[-p(x)y_2'(x) - q(x)y_2(x)] - [-p(x)y_1'(x) - q(x)y_1(x)]y_2(x) \\ &= -p(x)y_1(x)y_2'(x) - q(x)y_1(x)y_2(x) + p(x)y_1'(x)y_2(x) + q(x)y_1(x)y_2(x) \\ &= -p(x)y_1(x)y_2(x) + p(x)y_1'(x)y_2(x) \\ &= -p(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)] \\ &= -p(x)W(x) \end{aligned}$$

In other words, the Wronskian satisfies the first order linear equation.

$$W'(x) + p(x)W(x) = 0$$

This fact is known as Abel's theorem. We can easily solve it, and derive

$$W(x) = W(x_0)\exp \int_{x_0}^{x} p(t)\mathrm{d}t$$

the formula known as Abel's formula or Abel's identity. If the coefficient $p(x)$ is constant, then

$$W(x) = W(x_0)e^{(x-q_0)p}$$

An important consequence of Abel's formula is that the Wronskian of two solutions of ODE is either zero everywhere, or nowhere zero.

Example: We know that $y_1(x) = \cos\, x$ and $y_2(x) = \sin\, x$ are solutions to $y'' + y = 0$.

Solution: Since $p = 0$ in this case, in light of Abel's formula, the Wronskian $W(x)$ of y_1 and y_2 must be a. constant. We confirm it by explicit computation:

$W(x) = \cos x(\sin x)' - (\cos x)'\sin x = \cos^2 x + \sin^2 x = 1$

Example: The functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions to $y'' - 2y' + y = 0$. Since $p = -2$, we have $W(x) = ce^{2x}$ for some constant c. Explicit computation gives $W(x) = e^x(xe^x)' - (e^x)'e^x = e^x(e^x + xe^x) - xe^{2x} = e^{2x}$, so $c = 1$.