Group Theory

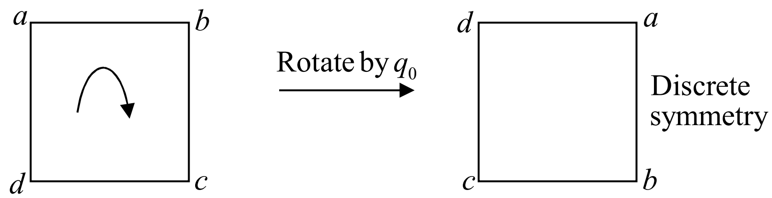
1. Introduction

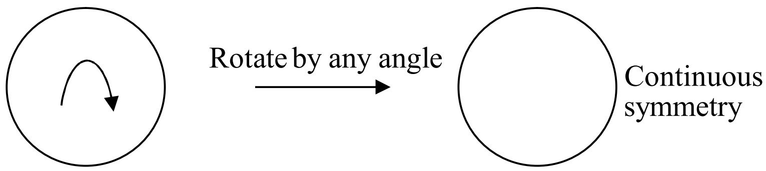
Group theory is in short, the mathematics of symmetries. Symmetries are very important in understanding or simplifying physics problems.

Symmetries have three properties which are key in understanding their mathematical structure; they are associative. There is an identity element and for every symmetry transformation, there is an inverse transformation that cancel it and bring us back to the original system. These three properties are what defines a “group” and the theory of these mathematical structure is “group theory”.

In summary using group theory we can exploit symmetry present in the physical system, to get some information about the physical system without actually solving the complex problem (e.g. solution Schrödinger equation).

Symmetries broadly can be classified as either continuous or discrete. In each case, the symmetry operations are represented by individual group elements. Discrete symmetry entails discrete groups, which may contain a finite number of elements. Continuous symmetries entail continuous (Lie) groups.





Group: A group G is defined as a set of objects or operations (e.g. relation or other transformation) called the elements of G, that may combined, by a procedure to be called, group multiplication and denoted by * to form a third element, in such a way that four conditions called group axioms are satisfied, namely:

1. Closure property: If $a, b \in G$ then

$$a \times b = c \in G$$

2. Associativity: if $a, b \,\&\, c \in G$, then

$$(a \times b) \times c = a \times (b \times c)$$

3. Unit element: There is an element e in G such that

$$ea = ae \;\; \forall \; a \in G$$

4. Inverse: For every element $a \in G$, there is a corresponding inverse element $a^{-1} \in G$ such that

$$a^{-1}a = aa^{-1} = e$$

Example: Let $S = \{1, -1, i, -1\}$ and group operation in multiplication

Check closure property	Associativity
$1 \times i = i$	$1 \times (-1 \times i) = -i$
$i \times (-i) = 1$	$(1 \times -1) \times i = -i$
$-1 \times -1 = 1$	$1 \times (-1 \times i)$

$i \times i = -1$ $\quad\quad = (1\times 1)\times 2$

Inverse	$1\times 1 = 1\sim$	inverse of 1 is 1
	$-1\times -1 = 1$	inverse of -1 is -1
	$i \times -i = 1$	inverse of i is $-i$
	$-i \times i = 1$	inverse of $-i$ is i

check

Example 2 Let $S = \{I, A\}$ matrix multiplication

Inverse $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Forms group under matrix multiplication

Closure $II = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad AI = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

Associativity: Matrix multiplication is always associative

Identity $IA = AI = A$

Inverse $AA = I$ $\quad\quad A$ is its own inverse

Example 3 Addition Module n

Show $z = \{0,1,2,3,4\}$ addition module 5 is group

Solution: Addition module 5 is define as

$a + 5b =$ least non-negative remainder when $a+2$ is divided by n

Closure $0+0=0$

$2+3=5=0$

$1+4=0=4+1$

$2+3=0=3+2$

$2+4=1=4+2$

$3+4=2=4+3$

Associativity is trivial

$2+(3+4)=2+7=9=4$

$(2+3)+4=5+4=9=4$

$2+(3+4)=(2+3)+4$

Identity element is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

Abelian group If group is commutative i.e.,

$\forall\ a,b\in G$

$a\times b=b\times a$

then group is called Abelian

Example: The set of all integers Z is a group if the group operation is taken to be the usual addition of integers. This group in abelian and has infinite number of elements

2. The set of all complex number C is a group under addition of complex numbers. This group is abelian and infinite

3. The set $C-\{0\}$ is an infinite abelian group under the usual multiplication of complex number

4. The set of all 2×2 matrices with complex entries is an infinite abelian group under matrix addition

Q1. Consider a group $G=Z$ (set of all integers) with the group operation $m\times n=m+n+1$ for $m,n\in G$ then the inverse of any element $P\in G$ is

(a) $-P-1$ (b) $\frac{1}{P}$ (c) $-P-2$ (d) $-P$

Solution: definition if identity element

$P\times e=P$

but $P\times e=P+e+1\Rightarrow P=P+e+1\Rightarrow e=-1$

now definition of inverse

$P\times P^{-1}=e=-1$

$P+P^{-1}+1=-1$

$P^{-1}=-P-Z$