Chapter 10

Numerical Technique

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Bisection Method

This method states that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and let its approximate value be given by $x_0 = \frac{(a+b)}{2}$.

If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x) = 0$. Otherwise, the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $[a_1, b_1]$ whose length is $|b-a|/2$. As before, this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval (which contains the root) is as small as desired, say ε. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n th step, the new interval will be $[a_n, b_n]$ of length $|b-a|/2^n$. We then have

$$\frac{|b-a|}{2^n} \leq \varepsilon$$

which gives on simplification

$$n \geq \frac{\log_e(|b-a|/\varepsilon)}{\log_e 2}$$

Above equation gives the number of iterations required to achieve an accuracy ε

For example, if $|b-a| = 1$ and $\varepsilon = 0.001$, then it can be seen that

$$n \geq 10$$

It can be easily programmed using the following computational steps:

1. Choose two real numbers a and b such that $f(a)f(b) < 0$.

2. Set $x_r = (a+b)/2$.

3. (a) If $f(a)f(x_r) < 0$, the root lies in the interval (a, x_r). Then, set $b = x_r$ and go to step 2 above.

(b) If $f(a)f(x_r) > 0$, the root lies in the interval (x_r, b). Then, set $a = x_r$ and go to step 2.

(c) If $f(a)f(x_r) = 0$, it means that x_r is a root of the equation $f(x) = 0$ and the computation may be terminated.

In practical problems, the roots may not be exact so that condition (c) above is never satisfied. In such a case, we need to adopt a criterion for deciding when to terminate the computations.

A convenient criterion is to compute the percentage error ε_r defined by

$$\varepsilon_r = \left| \frac{x_r' - x_r}{x_r'} \right| \times 100\%$$

where x_r' is the new value of x_r. The computations can be terminated when ε_r becomes less than a prescribed tolerance, say ε_p. In addition, the maximum number of iterations may also be specified in advance.

Example: Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution: Since $f(1)$ is negative and $f(2)$ positive, a root lies between 1 and 2 and, therefore, we take $x_0 = 3/2$. Then $f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$, which is positive. Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1+1.5}{2} = 1.25$$

We find $f(x_1) = -19/64$, which is negative. We, therefore, conclude that the root lies between 1.25 and 1.5. It follows that

$$x_2 = \frac{1.25+1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \quad x_4 = 1.34375, \quad x_5 = 1.328125, \text{ etc.}$$

Example: Find a real root of the equation $x^3 - 2x - 5 = 0$.
Solution: Let $f(x) = x^3 - 2x - 5$. Then f(2)=-1 and f(3)=16 .
Hence a root lies between 2 and 3 and we take

$$x_1 = \frac{2+3}{2} = 2.5$$

Since $f(x_1) = f(2.5) = 5.6250$, the root lies between 2 and 2.25. Hence

$$x_2 = \frac{2+2.5}{2} = 2.25$$

Now, $f(x_2) = 1.890625$, the root lies between 2 and 2.25. Therefore,

$$x_3 = \frac{2+2.25}{2} = 2.125$$

Since $f(x_3) = 0.3457$, the root lies between 2 and 2.125. Therefore,

$$x_4 = \frac{2+2.125}{2} = 2.0625$$

Proceeding in this way, we obtain the successive approximations:

$$\begin{aligned} &x_5 = 2.09375, \quad &x_6 = 2.10938, \quad &x_7 = 2.10156, \\ &x_8 = 2.09766, \quad &x_9 = 2.09570, \quad &x_{10} = 2.09473, \\ &x_{11} = 2.09424, \ldots \end{aligned}$$

We find

$$x_{11} - x_{10} = -0.0005,$$

and

$$\left| \frac{x_{11} - x_{10}}{x_{11}} \right| \times 100 = \frac{0.0005}{2.09424} \times 100 = 0.02\%$$

Hence a root, correct to three decimal places, is 2.094.