Tensor Analysis

12. Covariant Derivative

The covariant derivative of a tensor A_p with respect to x^{-4} is denoted by $A_{p \cdot A}$ and is defined by

$$A_{ps} = \frac{\partial A_p}{\partial e^q} - \left\{ \begin{matrix} 3 \\ pq \end{matrix} \right. \mid A_s$$

a covariant tensor of rank two.

The covariant derivative of a tensor A^p with respect to x^4 is denoted by A_{A^p} and is defined by

$$A^P = \frac{\mathrm{AA}^F}{\partial x^4} + \left\{ \begin{matrix} p \\ q8 \end{matrix} \right\} A^5$$

a mixed tensor of rank two.

For rectangular systems, the Christoffel symbols are zero and the covariant derivatives are the usual partial derivatives. Covariant derivatives of tensors are also tensors.

The above results can be extended to covariant derivatives of higher rank tensors. Thus,

$$\begin{aligned} A^{p_1\cdots p_m}_{r_1\cdots r_n,q} \equiv \; & \frac{\partial A^{p_1\cdots p_m}_{r_1\cdots r_n}}{\partial x^q} \\ & -\begin{Bmatrix} s \\ r_1 q \end{Bmatrix} A^{p_1\cdots p_m}_{s r_2\cdots r_n} - \begin{Bmatrix} s \\ r_2 q \end{Bmatrix} A^{p_1\cdots p_m}_{r_1 s r_3\cdots r_n} - \cdots - \begin{Bmatrix} s \\ r_n q \end{Bmatrix} A^{p_1\cdots p_m}_{r_1\cdots r_{n-1} s} \\ & + \begin{Bmatrix} p_1 \\ q s \end{Bmatrix} A^{s p_2\cdots p_m}_{r_1\cdots r_n} + \begin{Bmatrix} p_2 \\ q s \end{Bmatrix} A^{p_1 s p_3\cdots p_m}_{r_1\cdots r_n} + \cdots + \begin{Bmatrix} p_m \\ q s \end{Bmatrix} A^{p_1\cdots p_{m-1} s}_{r_1\cdots r_n} \end{aligned}$$

is the covariant derivative of Ap_1p_n with respect to x^q.

The rules of covariant differentiation for sums and products of tensors are the same as those for ordinary differentiation. In performing the differentiations, the tensors $\mathrm{g}_{\mathrm{p}}, \mathrm{g}^{2\psi}$, and δ^p_4 may be treated as constants since their covariant derivatives are zero. Since covariant derivatives express rates of change of physical quantities independent of any frames of reference, they are of great importance in expressing physical laws.