Introduction To Statistical Mechanics

2. Binomial Distribution and Random Walk

A drunk starts out from a lamppost located on a street. Each step he takes is of equal length l. The man is, however, so drunk that the direction of each step – whether it is to the right or to the left is completely independent of the preceding step. All one can say is that each time the man takes a step, the probability of its being to the right is p, while the probability of its being to the left is $q = 1 - p$. (In the simplest case $p = q$, but in general $p \neq q$. (For example, the street might be inclined with respect to the horizontal, so that a step downhill to the right is more likely than one uphill to the left.)

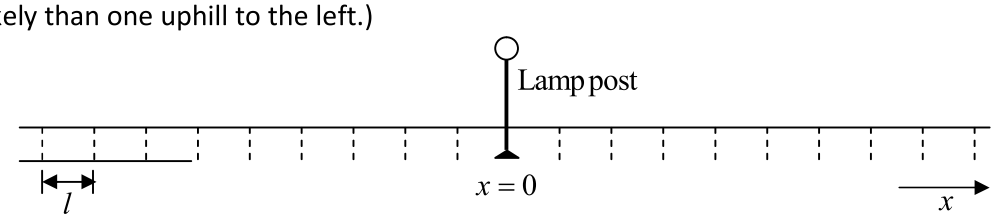


Fig. 1.1 Example of random walk in one dimension

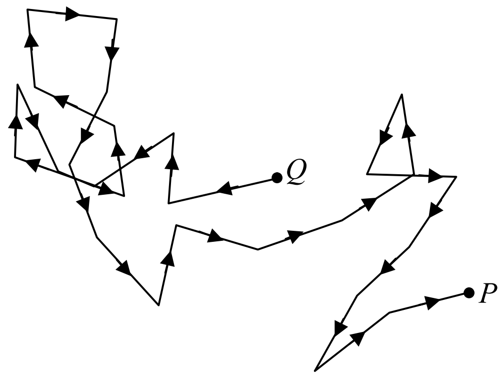


Fig. 1.2 Example of random walk in two dimensions

Choose the x-axis to lie along the street so that $x = 0$ is the position of the point of origin, the lamppost. Clearly, since each step is of length l, the location of the man along the x- axis must be of the form $x = ml$, where m is an integer (positive, negative, or zero).

Q: **After the man has taken N steps, what is the probability of his being located at the position $x = ml$?**

Consider few physical examples where this question is relevant and the same concept can be applied.

(a) **Magnetism:** An atom has a spin $\frac{1}{2}$ and a magnetic moment μ; in accordance with quantum mechanics, its spin can therefore point either "up" or "down" with respect to a given direction. If both these possibilities are equally likely, what is the net total magnetic moment of N such atoms?

(b) **Diffusion of a molecule in a gas:** A given molecule travels in three dimensions a mean distance l between collisions with other molecules. How far is it likely to have gone after N collisions?

(c) **Light intensity due to N incoherent light sources:** The light amplitude due to each source can be represented by a two-dimensional vector whose direction specifies the phase of the disturbance. Here the phases are random, and the resultant amplitude, which determines the total intensity of the light from all the sources, must be computed by statistical means.

The simple random walk problem in one dimension

Consider a particle performing successive steps, or displacements, in one dimension. After a total of N such steps, each of length l, the particle is located at

$$x = ml$$

Where, m is an integer lying between $-N \le m \le N$

We want to calculate the probability $P_N(m)$ of finding the particle at the position $x = ml$ after N such steps.

Let n_1 denote the number of steps to the right and n_2 the corresponding number of steps to the left. The total number of steps N is simply

$$N = n_1 + n_2$$

The net displacement (measured to the right in units of a step length) is given by

$$m = n_1 - n_2$$

If it is known that in some sequence of N steps the particle has taken n_1 steps to the right, then its net displacement from the origin is determined. Indeed, the preceding relations immediately yield

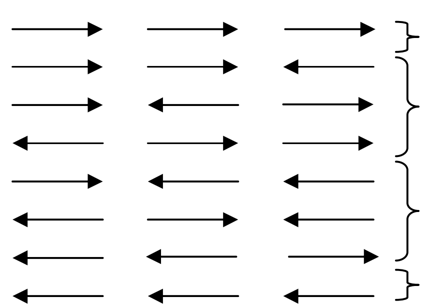
$$m = n_1 - n_2 = n_1 - (N - n_1) = 2n_1 - N$$

This shows that if N is odd, the possible values of m must also be odd. Conversely, if N is even, m must also be even.

Our fundamental assumption was that successive steps are statistically independent of each other. Thus, one can assert simply that, irrespective of past history, each step is characterized by the respective probabilities

$p =$ Probability that the step is to the right

and $q = 1 - p =$ probability that the step is to the left



R n_1	L n_2	m
3	0	3
2	1	1
1	2	−1
0	3	−3

Fig 1.3 Illustration showing the eight sequences of steps which are possible if the total number of steps is $N = 3$

Now, the probability of any one given sequence of n_1 steps to the right and n_2 steps to the left is given simply by multiplying the respective probabilities, i.e., by

$$\underbrace{pp...p}_{n_1 \text{ factors}}\underbrace{qq...q}_{n_2 \text{ factors}} = p^{n_1}q^{n_2}$$

But there are many different possible ways of taking N steps so that n_1 of them are to the right and n_2 are to the left (see illustration in figure). Indeed, the number of distinct possibilities (as shown below) is given by $\frac{N!}{n_1!n_2!}$

Hence the probability $W_N(n_1)$ of taking (in a total of N steps) n_1 steps to the right and $n_2 = N - n_1$ steps to the left, in any order, is obtained by multiplying the probability of this sequence by the number of possible sequences of such steps. This gives

$$\boxed{W_N(n_1) = \frac{N!}{n_1!n_2!}p^{n_1}q^{n_2}}$$

Example: Consider the simple illustration of Fig. 1.3 which shows the case of a total of $N = 3$ steps.

(a) How many number of ways are there that all three successive steps can be to the right?

(b) The corresponding probability $W(3)$ that all three steps are to the right is?

(c) On the other hand, the probability of a sequence of steps where two steps are to the right while the third step is to the left is?

Solution: (a) There is only one way.

(b) $p \cdot p \cdot p = p^3$.

(c) p^2q But there are three such possible sequences. Thus the total probability of occurrence of a situation where two steps are to the right and one is to the left is given by $3p^2q$.

Example: Consider the previous example in which three steps, there are $N=3$ possible events (or places) designated in Fig. 1.4 by B_1, B_2, B_3 and capable of being filled by the three particular steps labeled A_1, A_2, A_3. The event B_1 can occur in any of three ways, B_2 in any of two ways, and B_3 in only one way.

How many possible sequences of these three steps are there?

Solution: There are thus $3\times2\times1=3!=6$ possible sequences of these three steps.

But suppose that A_1 and A_2 denote both right steps $(n_1=2)$ while A_3 denotes a left step $(n_2=1)$. Then sequences of steps differing only by the two permutations of A_1 and A_2 are really identical. Thus one is left with only $\frac{6}{2}=3$ distinct sequences for which two steps are to the right and one step is to the left.

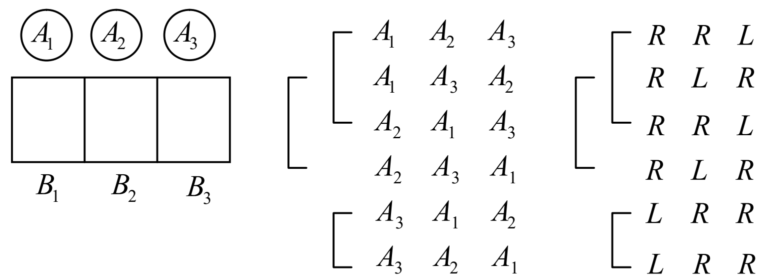


Figure: Diagram illustrating the problem of distributing three objects A_1, A_2, A_3 among three places B_1, B_2, B_3. The right part of the diagram lists the possible arrangements and indicates by brackets those arrangements which are identical when A_1 and A_2 are considered indistinguishable.