INTERPOLATION

Newton forward difference

Given the set of $(n+1)$ values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the n th degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots, n.$$

Since $y_n(x)$ is a polynomial of the n th degree, it may be written as

$$\left.\begin{aligned} y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \\ +a_3(x-x_0)(x-x_1)(x-x_2) + \cdots \\ +a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1}). \end{aligned}\right\}$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

$$a_0 = y_0; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; a_2 = \frac{\Delta^2 y_0}{h^2 2!}; a_3 = \frac{\Delta^3 y_0}{h^3 3!}; \cdots; a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, Then

$$\begin{aligned} y_n(x) = & y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\ & + \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!}\Delta^n y_0, \end{aligned}$$

which is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

Newton backward difference

Instead of assuming $y_n(x)$ as in above Eq., if we choose it in the form

$$\begin{aligned} y_n(x) = & a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) \\ & + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots \\ & + a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1) \end{aligned}$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_1, x_{n-1}, \ldots, x_2, x_1, x_0$, we obtain (after some simplification)

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n,$$

where $p = (x - x_n)/h$.

This is Newton's backward difference interpolation formula and it uses tabular values to the left of y_n. This formula is therefore useful for interpolation near the end of the tabular values.

Example: Find the cubic polynomial which takes the following values: $y(1) = 24, y(3) = 120, y(5) = 336$, and $y(7) = 720$. Hence, or otherwise, obtain the value of $y(8)$.

Solution: We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x-1)/2$. Substituting this value of p, we obtain

$$\begin{aligned} y(x) &= 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)}{2}(120) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{6}(48) \\ &= x^3 + 6x^2 + 11x + 6 \end{aligned}$$

To determine $y(8)$, we observe that $p = 7/2$. So

$$y(8) = 24 + \frac{7}{2}(96) + \frac{(7/2)(7/2-1)}{2}(120) + \frac{(7/2)(7/2-1)(7/2-2)}{6}(48) = 990.$$

Example: Find the missing term in the following table:

x	y
0	1
1	3
2	9
3	–
4	81

Explain why the result differs from $3^3 = 27$.

Solution: Since four points are given, the given data can be approximated by a third degree polynomial in x. Hence $\Delta^4 y_0 = 0$. Substituting $\Delta = E - 1$ and simplifying, we get

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

Since $E^r y_0 = y_r$, the above equation becomes

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

Substituting for y_0, y_1, y_2 and y_4 in the above, we obtain

$$y_3 = 31$$

The tabulated function is 3^x and the exact value of $y(3)$ is 27 . The error is due to the fact that the exponential function 3^x is approximated by means of a polynomial in x of degree 3.