Simpson's 1/3-rule

This rule is obtained by putting $n = 2$ in Eq.(1), i.e. by replacing the curve by $n/2$ arcs of second-degree polynomials or parabolas. We have then

$$\int_{x_0}^{x_2} ydx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_2}^{x_4} ydx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

and finally

$$\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

Summing up, we obtain

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + y_n\right]$$

which is known as Simpson's 1/3-rule. It should be noted that this rule requires the division of the whole range into an even number of sub intervals of width h.

The error in Simpson's rule is given by

$$\int_a^b ydx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + y_n\right.$$

$$= -\frac{b-a}{180}h^4 y^{\text{iv}}(\bar{x})$$

where $y^{\text{iv}}(\bar{x})$ is the largest value of the fourth derivatives.

Simpson's 3/8-rule

Setting $n = 3$ in Eq.(1), we observe that all the differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} ydx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

$$\int_{x_0}^{x_3} y dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly

$$\int_{x_3}^{x_6} y dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

and so on. Summing up all these, we obtain

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots$$

$$+2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

This rule, called Simpson's (3/8)-rule, is not so accurate as Simpson's rule, the dominant term in the error of this formula being $-(3/80)h^5 y^{\text{iv}}(\bar{x})$.

Example: A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines $x = 0$ and $x = 1$, and a curve through the points with the following coordinates:

x	y
0.00	1.0000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

Estimate the volume of the solid formed, giving the answer to three decimal places by Simpson's 1/3 rule

Solution: If V is the volume of the solid formed, then we know that

$$V = \pi \int_0^1 y^2 dx$$

Hence we need the values of y^2 and these are tabulated below, correct to four decimal places

x	y^2
0.00	1.0000
0.25	0.9793
0.50	0.9195
0.75	0.8261
1.00	0.7081

With $h = 0.25$, Simpson's rule gives

$$V = \frac{\pi(0.25)}{3}[1.0000 + 4(0.9793 + 0.8261) + 2(0.9195) + 0.7081] = 2.8192.$$

Example: Evaluate

$$I = \int_0^1 \frac{1}{1+x} dx$$

correct to three decimal places by both the trapezoidal and Simpson's rules with $h = 0.5, 0.25$ and 0.125 respectively.

Solution: (i) $h = 0.5$: The values of x and y are tabulated below:

x	y
0.0	1.0000
0.5	0.6667
1.0	0.5000

(a) Trapezoidal rule gives

$$I = \frac{1}{4}[1.0000 + 2(0.6667) + 0.5] = 0.70835$$

(b) Simpson's rule gives

$$I = \frac{1}{6}[1.0000 + 4(0.6667) + 0.5] = 0.6945$$

(ii) $h = 0.25$: The tabulated values of x and y are given below:

x	y
0.00	1.0000
0.25	0.8000
0.50	0.6667
0.75	0.5714
1.00	0.5000

(a) Trapezoidal rule gives

$$I = \frac{1}{8}[1.0 + 2(0.8000 + 0.6667 + 0.5714) + 0.5] = 0.6970.$$

(b) Simpson's rule gives

$$I = \frac{1}{12}[1.0 + 4(0.8000 + 0.5714) + 2(0.6667) + 0.5] = 0.6932$$