Tensor Analysis

9. Associated Tensors

Given a tensor, we can derive other tensors by raising or lowering indices. For example, given the tensor $A_{\rho p_0}$, by raising the index p, we obtain the tensor A^p, the dot indicating the original position of the moved index. By raising the index q also, we obtain A^{pq}. Where no confusion can arise, we shall often omit the dots; thus A^{py} can be written $A^{p\,\mathrm{s.}}$. These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{p4}. Thus, for example

$$A^p_{\cdot q} = g^{rp} A_{rq},\ A^{pq} = g^{rp} g^{sq} A_{rs},\ A^p_{\cdot rs} = g_{rq} A^{pq}_{\cdot s},\ A^{qm\cdot tk}_{\cdots n} = g^{pk} g_{sn} g^{rm} A^{q\cdot st}_{\cdot r \cdot p}$$

These become clear if we interpret multiplication by g'^p as meaning: let $r = p$ (or $p = r$) in whatever follows and raise this index. Similarly, we interpret multiplication by r as meaning: let $r = q$ (or $q = r$) in whatever follows and lower this index.

All tensors obtained from a given fensor by forming inner products with the metric tensor and its conjugate are called associated tensors of the given tensor. For example $A^{n\pi}$ and A_N are associated tensors, the first contravariant and the second covariant components. The relation between them is given by

$$A_p = g_{\mathrm{NQ}} A^v \text{ or } A^F = g^{Nv} A_q$$

For ractangular coordinates $g_p = 1$ if $p = q$, and 0 if $p \neq q$, so that $A_p = \mathrm{A}$, which explains why no distinction was made between contravariant and covariant components of a vector in earlier chapters.