Chapter 4

# Central Force and Kepler's System

## 1. Central Force

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

In central force potential $V$ is only function of $r$, a central force is always a conservative force; the magnitude $F$ of a central force can always be expressed as the derivative of a time-independent potential energy

$$\vec{\nabla}\times\vec{F}=\frac{1}{r\sin\theta}\left(\frac{\partial F}{\partial\phi}\right)\hat{\theta}-\frac{1}{r}\left(\frac{\partial F}{\partial\theta}\right)\hat{\phi}=0$$

And the force $F$ is defined as $F = -\frac{\partial V}{\partial r}\hat{r}$ (force is only in radial direction)

**External torque and angular momentum of system**

But for central force, $\tau = \vec{r} \times \vec{F}_r \Rightarrow \tau = r\hat{r} \times -\frac{\partial V}{\partial r}\hat{r} = 0$,

External torque $\tau = 0$, so angular momentum $\vec{J}$ is conserved.

Now, if we calculate $\vec{r} \cdot \vec{J} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0 \Rightarrow \vec{r} \perp \vec{J}$ , hence position vector $\vec{r}$ is perpendicular to angular momentum vector $\vec{J}$, hence $\vec{J}$ is conserved. Its magnitude and direction both are fixed, The central force $f(r)\hat{r}$ is along to $r$ and can exert no torque on the mass $m$. Hence the angular momentum $J$ of $m$ is constant. However, $J$ is fixed in space, and it follows that $r$ can only move in the plane perpendicular to $J$ through the origin.

**Equation of Motion for central force**

The equation of motion in polar coordinate is given by $m(\ddot{r} - r\dot{\theta}^2) = F_r$ and $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$

For central force problem $F(r) = -\frac{\partial V}{\partial r}$ and $F_\theta = 0$

$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}$ and $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

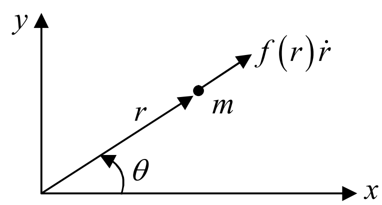


Figure 1

**Equation of Motion and Condition for Circular Orbit**

Condition of circular orbit from equation of motion in radial part $m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r}$

For circular orbit of radius $r_0$, $r = r_0$ and $\ddot{r} = 0$, so $-mr_0\dot{\theta}^2 = -\left.\frac{\partial V}{\partial r}\right|_{r=0}$

And $\dot{\theta} = \omega_0$ is identified as angular frequency in circular orbit.

$$-mr_0\dot{\theta}^2 = \left(-\frac{\partial V}{\partial r}\right)_{r=r_0} \Rightarrow mr_0\omega_0^2 = \left(\frac{\partial V}{\partial r}\right)_{r=r_0}$$

For circular orbit angular frequency $\omega_0$ is given by $\omega_0 = \sqrt{\frac{\left(\frac{\partial V}{\partial r}\right)_{r=r_0}}{m}}$

**Conservation of Angular Momentum and Areal Velocity**

$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$, but for central force, $F_\theta = 0 \Rightarrow m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \Rightarrow \frac{d(mr^2\dot{\theta})}{dt} = 0$

Angular momentum $= mr^2\dot{\theta} = J \Rightarrow \dot{\theta} = \dfrac{J}{mr^2}$ it is also seen $\vec{r}\cdot\vec{J} = 0$

So motion due to central force is confine into a plane and angular momentum $\vec{J}$ is perpendicular to that plane.

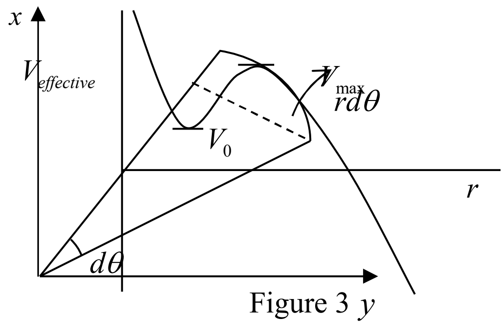


Figure 2

For the central force problem, now $\mathrm{A} = \dfrac{1}{2} r \cdot r d\theta$

Areal velocity $= \dfrac{d\mathrm{A}}{dt} = \dfrac{1}{2} r^2 \dfrac{d\theta}{dt} = \dfrac{1}{2} r^2 \dot{\theta}$

$\dfrac{d\mathrm{A}}{dt} = \dfrac{1}{2} r^2 \dot{\theta}$ . It is given that $\dot{\theta} = \dfrac{J}{mr^2}$ so $\dfrac{d\mathrm{A}}{dt} = \dfrac{J}{2m}$

Which means equal area will swept in equal time

**Total Energy of the System**

Hence total energy is not explicitly function of time $t$ so $\dfrac{\partial E}{\partial t} = 0$. One can conclude that total energy in central potential is constant.

$$E = \frac{1}{2}mv^2 + V(r) \text{ and Velocity } \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

So total energy, $E = \dfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = \dfrac{1}{2} m\dot{r}^2 + \dfrac{1}{2} mr^2\dot{\theta}^2 + V(r) \quad \left(\because\ \dot{\theta} = \dfrac{J}{mr^2}\right)$

$$= \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r), \quad r > 0$$

$$\Rightarrow E = \frac{1}{2} m\dot{r}^2 + V_{eff}$$

The sum rotational kinetic energy $\left(\dfrac{J^2}{2mr^2}\right)$ as a function of only $r$ and potential energy as a function of only $r$ i.e. $V(r)$ are identified as effective potential $V_{eff} = \dfrac{J^2}{2mr^2} + V(r)$

The concept of effective potential allow to two dimensional system in one system as $V_{effective}$ is only function of $r$.

**Analysis of effective potential**

The effective potential is define as $V_{eff} = \dfrac{J^2}{2mr^2} + V(r)$

The nature of orbit will dependent on nature of effective potential and total energy of a system which is shown in figure.

And lets discuss how variation of energy will leads to shape of orbit

**Case 1:** if energy $E < V_0$ there is turning point at $r = a$ the classical region as $r > a$, the particle is unbounded so nature of orbit may be either parabolic or hyperbolic

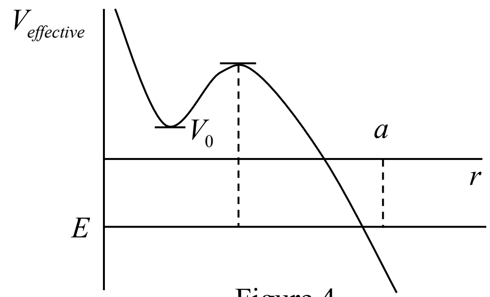


Figure 4

**Case 2:** If energy $E = V_0$ there $r = r_0$ is identified as stable equilibrium point because $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0$ and $\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0$ so possible orbit is circular in nature with radius $r_0$ for region, the angular frequency in circular orbit is $\omega_0 = \sqrt{\frac{\left(\frac{\partial V}{\partial r}\right)_{r=r_0}}{m}}$.

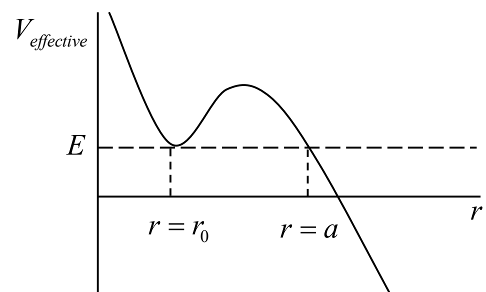


Figure 5

There is turning point at $r = a$. $r > a$ is also identified as classical region .The particle is unbounded so nature of orbit may be either parabolic or hyperbolic

**Case 3:** If energy $V_0 < E < V_{\max}$ the particle is bounded between turning point $a$ and $b$ and shape of orbit is elliptical in region.

Radius $r = r_0$ of circular orbit is also identified as stable equilibrium point, so $\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0} \geq 0$. Then new orbit is identified as elliptical orbit.

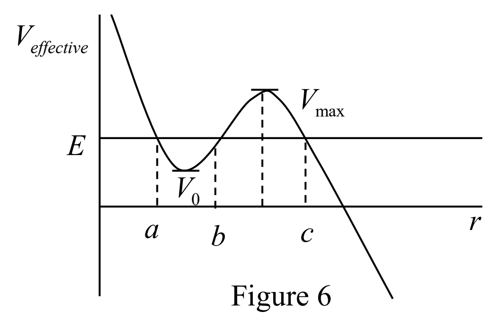


Figure 6

The angular frequency in new elliptical orbit is identified as oscillatory motion so for small oscillation the angular frequency is

$$\omega = \sqrt{\frac{\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}}{m}}$$

Another turning point is $r = c$, region $r > c$ is identified as classical region the particle is unbounded so nature of orbit may be either parabolic or hyperbolic

**Case 4:** $E = V_{\max}$ hence $r = b$ is unstable equilibrium point. , the orbit can be unstable circular of radius $b$ .there is another possible shape as it is elliptical between turning point $a$ and unstable equilibrium point $b$ .unbounded orbit is also possible for $r > b$

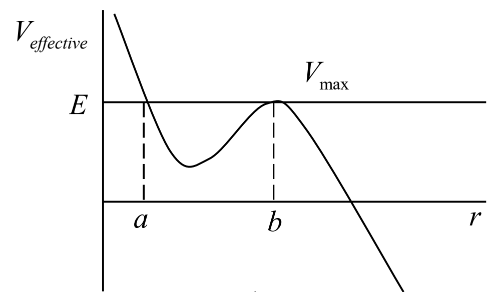


Figure 7

**Case 5:** $E > V_{\max}$ there is only one turning point $r = a$ and region $r > a$ is identified as classical region the particle is unbounded so nature of orbit may be either parabolic or hyperbolic

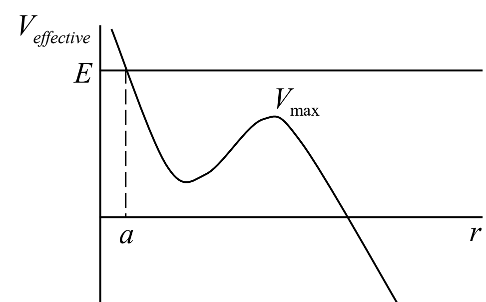


Figure 8

**Differential Equation of Orbit**

From equation of motion in radial part, $m(\ddot{r} - r\dot{\theta}^2) = f(r) \Rightarrow \frac{md^2r}{dt^2} - \frac{J^2}{mr^3} = f(r)$

where $J = mr^2\dot{\theta} \Rightarrow d\theta = \frac{J}{mr^2}dt \Rightarrow \frac{d}{dt} = \frac{J}{mr^2}\frac{d}{d\theta}$

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right) = \left(\frac{J}{mr^2}\right)\frac{d}{d\theta}\left(\frac{J}{mr^2}\right)\frac{d}{d\theta} \Rightarrow \frac{d^2r}{dt^2} = -\frac{J^2}{m^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

Put the value $\frac{d^2r}{dt^2} = -\frac{J^2}{m^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$ in equation $\frac{md^2r}{dt^2} - \frac{J^2}{mr^3} = f(r)$

$$\Rightarrow -\frac{J^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = f(r) \text{ Put } \frac{1}{r} = u \Rightarrow -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right)$$

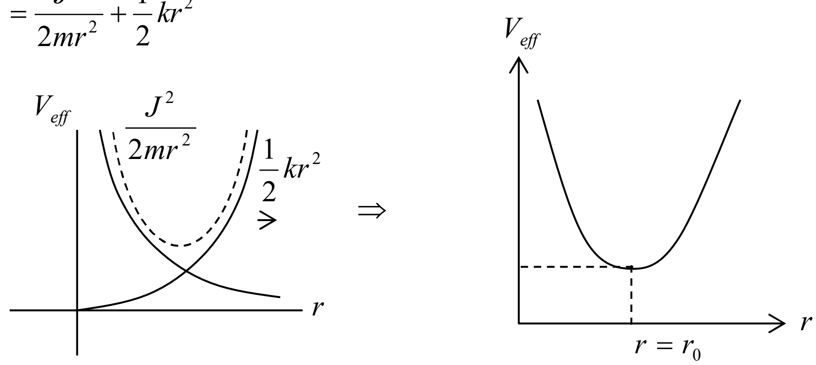
**Example:** Consider that the motion of a particle of mass $m$ in the potential field $V(r)=\frac{kr^2}{2}$. If $l$ is angular momentum,

(a) What is effective potential ($V_{eff}$) of the system. plot $V_{eff}$ vs $r$

(b) Find value of energy such that motion is circular in nature.

(c) If particle is slightly disturbed from circular orbit such that its angular remain constant. What will nature of new orbit? Find the angular frequency of new orbit in term of $m, l, k$.

**Solution:** (a) $V_{eff}=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2$




(b) $\frac{dV_{eff}}{dr}=-\frac{J^2}{mr^3}+kr=0$ at $r=r_0$ so $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ and $J=m\omega_0 r_0^2$

For circular motion, $m\omega_0^2 r_0=kr_0$, where $r_0$ is radius of circle $\omega_0=\sqrt{\frac{k}{m}}$

Total energy, $E=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2=\frac{mkr_0^4}{2mr_0^2}+\frac{1}{2}kr_0^2$ , $E=kr_0^2$ put $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ , $E=J\sqrt{\frac{k}{m}}$

(c) orbit is elliptical in nature

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}=\frac{3J^2}{mr^4}+k=\frac{3J^2}{m\left(\frac{J^2}{mk}\right)}+k=4k$$

$$\omega=\sqrt{\frac{\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}}{m}}=\sqrt{\frac{4k}{m}}=2\sqrt{\frac{k}{m}}=2\omega_0$$

**Example:** A particle of mass $m$ moves under the influence of an attractive central force $f(r)$.

(a) What is condition that orbit is circular in nature if $J$ is the angular momentum of particle

(b) If force is in form of $f(r)=\frac{-k}{r^n}$ determine the maximum value of n for which the circular orbit can be stable.

**Solution: (**a) if $V_{eff}=\frac{J^2}{2mr^2}+V(r)$, for circular stable orbit $\frac{\partial V_{eff}}{\partial r}=0$ and $\frac{\partial^2 V_{eff}}{\partial r^2}>0$

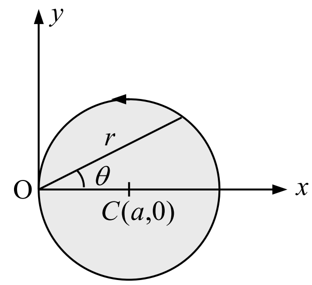
(b) $f(r)=\frac{-k}{r^n}$, for circular motion $\frac{\partial V_{eff}}{\partial r}=0\Rightarrow -\frac{J^2}{mr^3}+\frac{\partial V}{\partial r}=0$

It is given $\frac{\partial V}{\partial r}=-f(r)$ if $f(r)=\frac{-k}{r^n}$ $\Rightarrow \frac{\partial V}{\partial r}=\frac{k}{r^n}$

$$-\frac{J^2}{mr^3}+\frac{k}{r^n}=0\Rightarrow \frac{k}{r^n}=\frac{J^2}{mr^3}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2}>0\Rightarrow \frac{3J^2}{mr^4}-\frac{nk}{r^{n+1}}>0\Rightarrow \frac{3J^2}{mr^4}-\frac{n}{r}.\frac{J^2}{mr^3}>0 \text{ so } n<3$$

**Example:** A particle of mass $m$ and angular momentum $l$ is moving under the action of a central force $f(r)$ along a circular path of radius $a$ as shown in the figure. The force centre $O$ lies on the orbit.



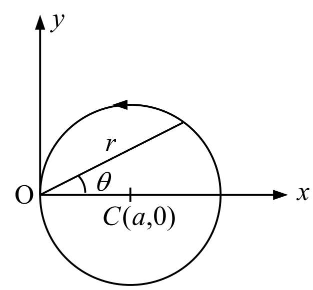

(a) Given the orbit equation in a central field motion.

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2u^2}f\text{, where } u=\frac{1}{r}$$

Determine the form of force in terms of $l, m, a$ and $r$.

(b) Calculate the total energy of the particle assuming that the potential energy $V(r)\to 0$ as $r\to\infty$.

**Solution:** (a) from the figure, $r=2a\cos\theta$, $\frac{1}{r}=\frac{\sec\theta}{2a}$




$$-\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right]=f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{(2a)^2m}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)+\frac{\sec\theta}{2a}\right]=f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{(2a)^2m}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta+\sec\theta\right)\right]=f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^3\theta}{2a\cdot 4a^2m}\left[\tan^2\theta+\sec^2\theta+1\right]=f\left(\frac{1}{u}\right)\Rightarrow-\frac{2J^2\sec^5\theta}{2a\cdot 4a^2m}=f\left(\frac{1}{u}\right)\Rightarrow f(r)\propto\frac{1}{r^5}$$

(b) $E=\frac{m\dot{r}^2}{2}+\frac{J^2}{2mr^2}+V(r)$ , $r\to\infty$, $V(r)\to 0$, $\frac{J^2}{2mr^2}\to 0$, as $r\to\infty$

$E=\frac{m\dot{r}^2}{2}$ and $r=2a\cos\theta$ and $\dot{r}=-2a\sin\theta\dot{\theta}$, $\dot{\theta}=\frac{J^2}{mr^2}$ as $r\to\infty$

Hence, $\dot{\theta}=\frac{J^2}{mr^2}\to 0$ so $\dot{r}\to 0$ so $E=0$