Chapter 5

Centre of Mass and Moment of Inertia

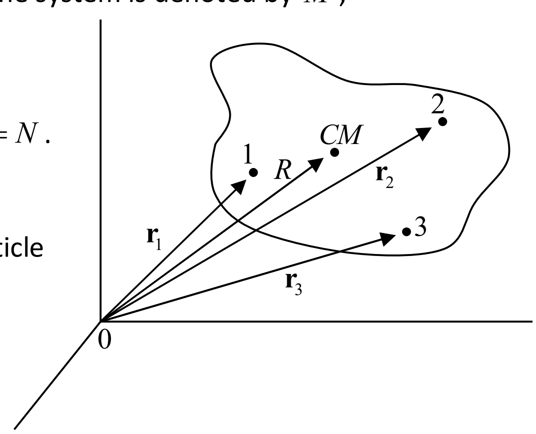
1. Centre of Mass

Consider a system composed of N particles with each particle's mass described by m_i, where i is an index from $i=1$ to $i=N$. The total mass of the system is denoted by M,

$$M=\sum_i m_i$$

where the summation over i runs from $i=1$ to $i=N$.

Such a system is displayed in figure given.



If the vector connecting the origin with the i^{th} particle is $\mathbf{r}_i$ then the vector defining the position of the system's center of mass is $\vec{\mathbf{R}}_{\text{C.M}}=\dfrac{1}{M}\sum_i m_i\vec{\mathbf{r}}_i$

The position of the centre of mass is defined by $R_{CM} \equiv \frac{\sum m_i r_i}{M}$

For N number of mass elements, if r_j is the position of the j^{th} element, and m_j is its mass, then the center of mass is defined as $\vec{R} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j$

Example: The position vector of three particles of mass $m_1 = 1kg, m_2 = 2kg$ and $m_3 = 3kg$ are $\vec{r}_1 = (\hat{i} + 4\hat{j} + \hat{k})m$, $\vec{r}_2 = (\hat{i} + \hat{j} + \hat{k})m$ and $\vec{r}_3 = (2\hat{i} - \hat{j} - 2\hat{k})m$ respectively. Find the position vector of their centre of mass.

Solution: The position vector of COM of the three particles will be given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

Substituting the values, we get

$$\vec{r}_{COM} = \frac{(1)(\hat{i} + 4\hat{j} + \hat{k}) + (2)(\hat{i} + \hat{j} + \hat{k}) + (3)(2\hat{i} - \hat{j} - 2\hat{k})}{1+2+3} = \frac{9\hat{i} + 3\hat{j} - 3\hat{k}}{6}$$

$$\vec{r}_{COM} = \frac{1}{2}(3\hat{i} + \hat{j} - \hat{k})m$$

Example: Four particles of mass $1kg, 2kg, 3kg$ and $4kg$ are placed at the four vertices A, B, C and D of a square of side $1m$. Find the position of centre of mass of the particles.

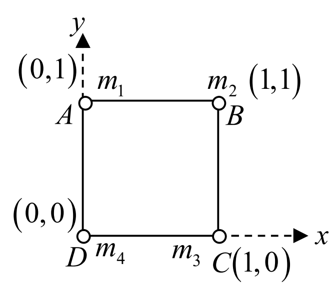
Solution: Assuming D as the origin, DC as x - axis and DA as y - axis, we have

$$m_1 = 1kg, (x_1, y_1) = (0, 1m)$$

$$m_2 = 2kg, (x_2, y_2) = (1m, 1m)$$

$$m_3 = 3kg, (x_3, y_3) = (1m, 0)$$

$$m_4 = 4kg, (x_4, y_4) = (0, 0)$$



and co-ordinates of their COM are

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4} = \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1+2+3+4} = \frac{5}{10} = \frac{1}{2}m = 0.5m$$

Similarly,

$$y_{CM} = x_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4} = \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1+2+3+4} = \frac{3}{10}m = 0.3m$$

$$\therefore \qquad (x_{COM}\, y_{COM}) = (0.5m, 0.3m)$$