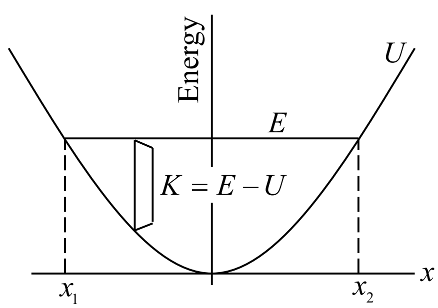
Chapter 1

Stability Analysis and Phase Diagram

1. Energy Diagrams

We can often find the most interesting features of the motion of a one dimensional system by using an energy diagram, in which the total energy E and the potential energy U are plotted as functions of position. The kinetic energy $K = E - U$ is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where $U \leq E$.

Energy Diagram of Bounded Motion



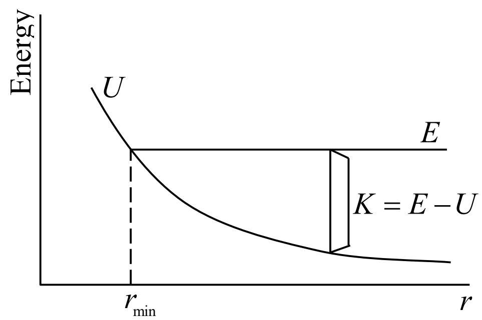
Here is the energy diagram for a harmonic oscillator. The potential energy $U = kx^2/2$ is a parabola centered at the origin. Since the total energy is constant for a conservative system, E is represented by a horizontal straight line. Motion is limited to the shaded region where $E \geq U$; the limits of the motion, x_1 and x_2 in the sketch, are sometimes called the turning points.

Here is what the diagram tells us. The kinetic energy, $K = E - U$ is greatest at the origin. As the particle flies past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points x_1, x_2. The particle then moves toward the origin with increasing kinetic energy and the cycle is repeated.

The harmonic oscillator provides a good example of bounded motion. As E increases, the turning points move farther and farther off, but the particle can never move away freely. If E decreased, the amplitude of motion decreases, until finally for $E = 0$ the particle lies at rest at $x = 0$.

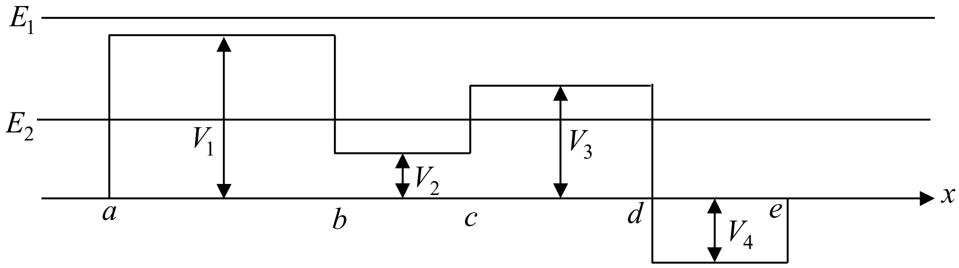
Energy Diagram of Unbounded Motion

The pot $U = A/r$, where A is positive. There is a distance of closest approach $r_{\min}$, as shown in the diagram, but the motion is not bounded for large r since U decreases with distance. If the particle is shot toward the origin, it gradually losses kinetic energy until it comes momentarily to rest at $r_{\min}$. The motion then reverses and the particle moves back towards infinity. The final and initial speeds at any point are identical; the collision merely reverses the velocity.



For positive energy, $E > 0$, the motion is unbounded, and the atoms are free to fly apart. As the diagram indicates, the distance of closest approach, $r_{\min}$, does not change appreciably as E is increased. The kinetic energy will be zero at $r_{\min}$ and as r increases the potential energy decreases, but kinetic energy $K = E - U$ will increase sharply.

Example:



In the above figure, potential energy in different regions are given, where

$$V(x)=\begin{cases} V_1 = 8J, & a<x<b \\ V_2 = 3J, & b<x<c \end{cases} \quad \text{and} \quad V(x)=\begin{cases} V_3 = 6J, & c<x<d \\ V_4 = -4J, & d<x<e \end{cases}$$

The potential is assumed to be zero in all other regions.

(a) What will be kinetic energy in all region if total energy E is $10J$?

Solution: If 'T' is kinetic energy and 'V' is potential energy, then total energy $E=T+V$, so kinetic energy is $T=E-V$.

For total energy $E=E_1$ all regions are classical allowed region.

So, in region $x<a$, $V(x)=0$, so $T=10-0=10J$

In region $a<x<b$, $V(x)=V_1=8J$ so $T=10-8=2J$

In region $b<x<c$, $V(x)=V_2=3J$ so $T=10-3=7J$

In region $c<x<d$, $V(x)=V_3=6J$ so $T=10-6=4J$

In region $d<x<e$, $V(x)=V_4=-4$ so $T=10-(-4)=14J$

In region $e<x$, $V(x)=0$ so $T=10-0=10J$

(b) What will be kinetic energy in all regions, if total energy E is $5J$?

Solution: If T is kinetic energy and V is potential energy then total energy $E=T+V$, so kinetic energy is $T=E-V$

So, in region $x<a$, $V(x)=0$ so, $T=5-0=5J$

In region $a<x<b$, $V(x)=V_1=8J$ hence $V_1>E_2$ so, $T=0$ (classical forbidden region)

In region $b<x<c$, $V(x)=V_2=3J$ so, $T=5-3=2J$

In region $c<x<d$, $V(x)=V_3=6J \Rightarrow V_3>E_2$ so, $T=0$ (classical forbidden region)

In region $d<x<e$, $V(x)=V_4=-4$ so, $T=5-(-4)=9J$

In region $e<x$, $V(x)=0$ so, $T=5-0=5J$