Chapter 3

Equation of Motion and Variable Mass

1. Equation of Motion

According to Newton's law of motion Force acting on a body $F = ma$

where acceleration a is given by $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \Rightarrow a = \frac{d^2x}{dt^2}$

If velocity is only function of position One can also represent acceleration as

$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt} \Rightarrow a = \frac{dv}{dx} \times \frac{dx}{dt} \Rightarrow a = v\frac{dv}{dx} \text{ where } v = \frac{dx}{dt}$$

to be used when v is a function of x only and $v = \frac{dx}{dt}$, $a = \frac{d^2x}{dt^2}$

Example: An object moving with a speed of $6.25\,m/s$ is decelerated at a rate given by $\frac{dv}{dt} = -k\sqrt{v}$ where $k = 2.5$ and v is the instantaneous speed. Find the time taken by the object to come to rest

Solution: $\frac{dv}{dt} = -k\sqrt{v} \Rightarrow \frac{dv}{\sqrt{v}} = -kdt$

Integrating $= -\int_0^t kdt = \int_{v=6.25}^{0} v^{-\frac{1}{2}} dv$ (at $t = 0,\ v = 6.25\,m/s$, at $t = t, v = 0$)

or $= -k[t]_0^t = \left[2\sqrt{v}\right]_{6.25}^{0}$ Putting $k = 2.5$, $t = 2\sec$

Example: Time and distance of an object are related as $t = \alpha x^2 + \beta x$, here α and β are constants. The retardation is given by kv^n. Find the values of k and n.

Solution: $t = \alpha x^2 + \beta x$

Differentiating wrt t, $\frac{dt}{dt} = \alpha \times 2x\frac{dx}{dt} + \beta\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{2\alpha x + \beta}$ and $a = \frac{d^2x}{dt^2} = -\frac{2\alpha\frac{dx}{dt}}{(2\alpha x + \beta)^2}$

$\Rightarrow a = -2\alpha v \times v^2 = -2\alpha v^3$. So $k = -2\alpha,\ n = 3$

Example: The acceleration experienced by a boat after the engine is cut off, is given by $\frac{dv}{dt} = -kv^3$, k is a constant. If v_0 is the magnitude of velocity at cut off, find the velocity after time t from cut off.

Solution: $\frac{dv}{dt} = -kv^3 \Rightarrow v^{-3}dv = -kdt$

$$\int_{v_0}^{v} v^{-3}dv = -\int_0^t kdt \Rightarrow -\frac{1}{2}\left(\frac{1}{v^2} - \frac{1}{v_0^2}\right) = -kt \Rightarrow v = \frac{v_0}{\sqrt{1 + 2ktv_0^2}}$$

Example: Displacement of a body is given by $s \propto t^3$, t is the time elapsed

(a) Find the relation between velocity v and displacement s

(b) Find the relation between acceleration a and displacement s

Solution: (a) As $s = kt^3$

$$s = kt^3 \Rightarrow v = \frac{ds}{dt} = 3kt^2 \Rightarrow t = \left(\frac{v}{3k}\right)^{1/2}$$

$$s = k \times \left(\frac{v}{3k}\right)^{1/2} = \left(\frac{1}{3k}\right)^{1/2} \times v^{1/2} \Rightarrow v \propto s^2$$

(b) $s = kt^3 \Rightarrow v = \frac{ds}{dt} = 3kt^2$ and acceleration $a = \frac{dv}{dt} \Rightarrow a = 6kt \Rightarrow t = \frac{a}{6k}$

$$s = kt^3 \Rightarrow s = k\left(\frac{a}{6k}\right)^3 \Rightarrow s = \frac{a^3}{216k^2} \Rightarrow s \propto a^3 \Rightarrow a \propto s^{1/3}$$

Example: A particle starts from rest moves with a velocity given by $v = kt$ (here $k = 2m/s^2$)

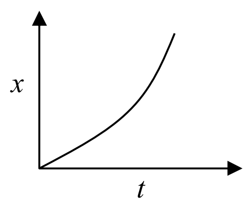
(a) Find the distance covered by the particle in first 3 sec.

(b) Draw distance versus time $x - t$, $x - v$ and $x - a$ graphs

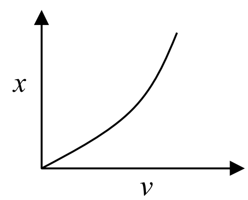
Solution: (a) $\frac{dx}{dt} = kt \Rightarrow \int_0^x dx = \int_0^t ktdt \; k\int_0^t tdt \Rightarrow x = \frac{1}{2}kt^2 = \frac{1}{2} \times 2 \times 3 = 9\,m$

(b) $x \propto t^2$ $x = kt^2$

$v = \frac{ds}{dt} = 2kt$

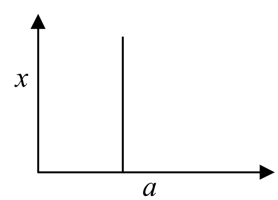


$x = k\left(\frac{v}{2k}\right)^2 = \frac{v^2}{4k} \Rightarrow x \propto v^2$



$a = \frac{dv}{dt} = 2k \Rightarrow a = 2k \times x^0$

$a \propto x^0$ (a is constant)



Example: We want to find the velocity of a falling parachutist as a function of time and are particularly interested in the constant limiting velocity, v_0, that comes about by air drag, taken to be quadratic, $-bv^2$ and opposing the force of the gravitational attraction, mg, of the Earth on the parachutist. We choose a coordinate system in which the positive direction is downward so that the gravitational force is positive. For simplicity we assume that the parachute opens immediately, that is, at time $t=0$, where $v(t)=0$, our initial condition.

(a) Write down equation of motion

(b) Find terminal velocity v_0

(c) if v_0 is terminal velocity then find the velocity after time t

Solution: (a) $m\frac{dv}{dt}=mg-bv^2$ (i)

where m includes the mass of the parachute.

(b) The terminal velocity, v_0 can be found from the equation of motion as $t\to\infty$, when there is no acceleration, $\dot{v}=0 \quad \Rightarrow bv_0^2=mg$ or $v_0=\sqrt{\frac{mg}{b}}$

(c) It simplifies further work to rewrite equation (i) as

$$\frac{m}{b}\dot{v}=v_0^2-v^2$$

This equation is separable and we write it in the form

$\frac{dv}{v_0^2-v^2}=\frac{b}{m}dt$ .....(2) Using partial fractions to write $\left(\frac{1}{v_0^2-v^2}=\frac{1}{2v_0}\left(\frac{1}{v+v_0^2}-\frac{1}{v-v_0}\right)\right)$

It is straightforward to integrate both sides of equation (2) (the left-hand side from $v=0$ to v, the right-hand side from $t=0$ to t), yielding $\frac{1}{2v_0}\ln\frac{v_0+v}{v_0-v}=\frac{b}{m}t$

Solving for the velocity, we have $v=\frac{e^{2t/T}-1}{e^{2t/T}+1}v_0=v_0\frac{\sinh(t/T)}{\cosh(t/T)}=v_0\tanh\frac{t}{T}$

where $T=\sqrt{m/gb}$ is the time constant governing the asymptotic approach of the velocity to its limiting value, v_0.