Chapter 4

Central Force and Kepler's System

2. Kepler's Potential

The Kepler's potential is given by $V(r) = -\frac{k}{r}$ with $k > 0$ which is central potential. If Particle of mass m and angular momentum J is interact with potential, then effective potential is given by $V_{effective} = \frac{J^2}{2mr^2} - \frac{k}{r}$.

If one will plot $V_{effective}$ against r

Analysis of $V_{effective} = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow \frac{\partial V_{effective}}{\partial r} = 0 \Rightarrow -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0 \Rightarrow r_0 = \frac{J^2}{mk}$

$\frac{\partial^2 V_{effective}}{\partial r^2} = \frac{3J^2}{mr^4} - \frac{2k}{r^3}$ the value of $\frac{\partial^2 V_{effective}}{\partial r^2}$ at $r = r_0 = \frac{J^2}{mk}$ is $\frac{m^3k^4}{J^6} > 0$ which is minima of the $V_{effective}$.

The Analysis of $V_{effective}$ for Kepler's law

Case 1 for $r = r_0 = \frac{J^2}{mk}$ total energy $E_0 = -\frac{mk^2}{2J^2}$ the motion of path is elliptical

Case 2 $E_0 < E < 0$ the particle is bounded between turning point $a < r < b$ and the motion is equivalent to small oscillation about stable point $r = r_0$ and shape of orbit is elliptical in nature.

Case 3 $E = 0$ the particle is unbounded where $r = c$ is turning point.

Case 4 $E > 0$ the particle is unbounded where $r = d$ is turning point.

Equation of orbit for Kepler's

Equation of motion is given by $m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2}$ put $\dot{\theta} = \frac{J}{mr^2} \Rightarrow m\ddot{r} - \frac{J^2}{mr^3} = -\frac{k}{r^2}$

Equation of orbit is given by $\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right)$

$$f(r) = -\frac{k}{r^2}, \quad f\left(\frac{1}{u}\right) = -ku^2 \Rightarrow \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = +ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{ku^2m}{J^2u^2}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{km}{J^2}\right) = 0 \text{ put } u - \frac{km}{J^2} = y \text{ so } \frac{d^2u}{d\theta^2} = \frac{d^2y}{d\theta^2}$$

The equation reduce to $\frac{d^2y}{d\theta^2} + y = 0$

The solution of equation reduce to $y = A\cos\theta \; u - \frac{km}{J^2} = A\cos\theta \Rightarrow u = \frac{km}{J^2} + A\cos\theta$

$$\frac{1}{r} = \frac{km}{J^2} + A\cos\theta \Rightarrow \frac{\frac{J^2}{km}}{r} = 1 + \left(\frac{AJ^2}{km}\right)\cos\theta$$

Put $\frac{J^2}{km} = l$ and $e = \frac{AJ^2}{km}$, then equation reduce to $\frac{l}{r} = 1 + e\cos\theta$, which is equation of conics where l is latus rectum and e is eccentricity.

In a central force potential which is interacting with potential $V(r) = -\frac{k}{r}$ can be any conics section depending on eccentricity e.

Relationship between Energy and Eccentricity

For central potential $V(r) = -\frac{k}{r}$, the solution of orbit is $\frac{l}{r} = 1 + e\cos\theta$ with $l = \frac{J^2}{km}$

The energy is given by $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{k}{r}$

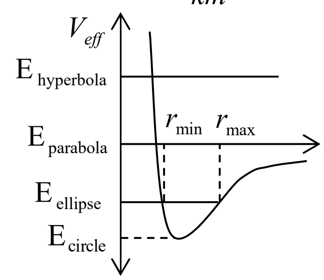

Figure 9

The solution of orbit is $\frac{l}{r} = 1 + e\cos\theta$ with $l = \frac{J^2}{km}$

So $\frac{-l}{r^2}\dot{r} = -e\sin\theta\dot{\theta}$ where $\dot{\theta} = \frac{J}{mr^2}$ so $\dot{r} = \frac{eJ\sin\theta}{ml}$,

After putting the value of $\frac{l}{r} = 1 + e\cos\theta$ and $\dot{r} = \frac{eJ\sin\theta}{ml}$ with $l = \frac{J^2}{km}$ in equation of energy,

one will get $e = \sqrt{1 + \frac{2EJ^2}{mk^2}}$

the condition on energy for possible nature of orbit for potential

$E > 0$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$E < 0$	$e < 1$	Ellipse
$E = -\frac{mk^2}{2J^2}$	$e = 0$	Circle