Chapter 5

Centre of Mass and Moment of Inertia

2. Rigid Body Dynamics

A rigid body is defined as system of particles in which the distance between any two particles remains fixed throughout the motion.

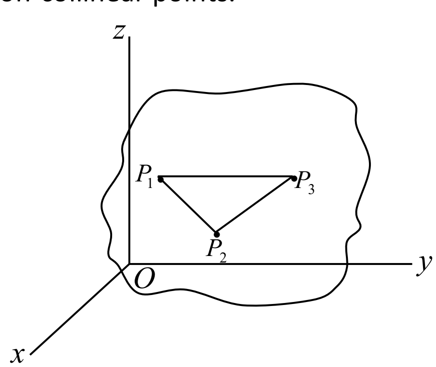
Degree of Freedom of Rigid Body

To define rigid body, there must be minimum 3 non-collinear points.

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ are three non-collinear points.

So, equation of constrained is

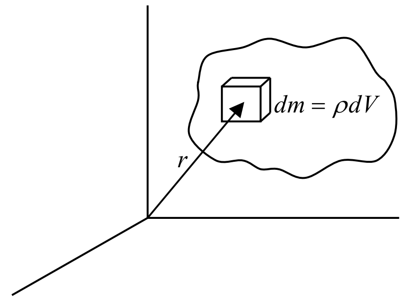
$$r_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = c_1$$
$$r_{23} = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = c_2$$
$$r_{13} = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = c_3$$
$$Dof = 3N - k$$
$$3\times 3 - 3 = 6$$



So there is six degree of freedom for rigid body

Center of Mass of Continuous System

The result is not rigorous, since the mass elements are not true particles. However, in the limit where N approaches infinity, the size of each element approaches zero and the approximation becomes exact.



$\vec{R} = \lim_{N\to\infty} \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$. This limiting process defines an integral.

Formally $\lim_{N\to\infty} \sum_{j=1}^{\infty} m_j r_j = \int r\, dm$, where dm is a differential mass element. Then,

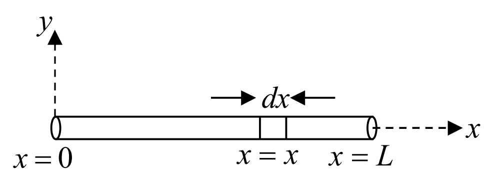
$\vec{R} = \frac{1}{M}\int \vec{r}\, dm \Rightarrow X_{CM} = \frac{1}{M}\int x dm, Y_{CM} = \frac{1}{M}\int y dm, Z_{CM} = \frac{1}{M}\int z dm$.

To visualize this integral, think of dm as the mass in an element of volume dV located at position r. If the mass density at the element is ρ, then $dm = \rho\, dV$ and $\vec{R} = \frac{1}{M}\int \vec{r} \rho dV$. This integral is called a volume integral.

Example: A rod of length L is placed along the x- axis between $x = 0$ and $x = L$. The linear density (mass/length) λ of the rod varies with the distance x from the origin as $\lambda = Rx$. Here, R is a positive constant. Find the position of centre of mass of this rod.

Solution: Mass of element dx situated at $x = x$ is

$$dm = \lambda dx = Rx\, dx$$



The COM of the element has coordinates $(x, 0, 0)$.

Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x dm}{\int_0^L dm} = \frac{\int_0^L (x)(Rx) dx}{\int_0^L (Rx) dx} = \frac{R\int_0^L x^2 dx}{R\int_0^L x dx} = \frac{\left[\left(\frac{x^2}{3}\right)\right]_0^L}{\left[\left(\frac{x^2}{2}\right)\right]_0^L} = \frac{2L}{3}$$

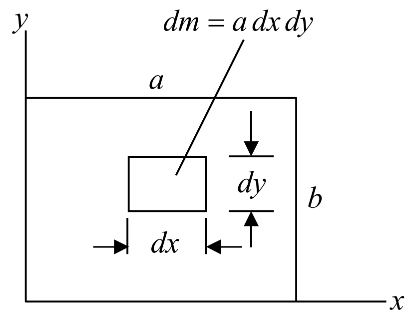
The y-coordinate of COM of the rod is $y_{COM} = \frac{\int y dm}{\int dm} = 0$ (as $y = 0$)

Similarly, $z_{COM} = 0$ Hence, the centre of mass of the rod lies at $\left[\frac{2L}{3}, 0, 0\right]$

Example: Find the center of mass of a thin rectangular plate with sides of length a and b, whose mass per unit area σ varies in the following fashion:

$\sigma = \sigma_0 \left(xy / ab \right)$, where σ_0 is a constant.

Solution: $\vec{R} = \frac{1}{M} \iint \left(x\hat{i} + y\hat{j} \right) \sigma dx\, dy$,



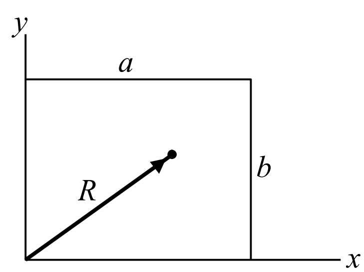
We find M, the mass of the plate, as follows:

$$M = \int_0^b \int_0^a \sigma dx\, dy = \int_0^b \int_0^a \sigma_0 \frac{x}{a} \frac{y}{b} dx\, dy$$

We first integrate over x, treating y as a constant.

$$M = \int_0^b \left(\int_0^a \sigma_0 \frac{x}{a} \frac{y}{b} dx \right) dy = \int_0^b \left(\sigma_0 \frac{y}{b} \frac{x^2}{2a} \bigg|_{x=0}^{x=a} \right) dy$$

$$= \int_0^b \sigma_0 \frac{y}{b} \frac{a}{2} dy = \frac{\sigma_0 a}{2} \frac{y^2}{2b} \bigg|_{y=0}^{y=b} = \frac{1}{4} \sigma_0 ab$$



The x component of R is

$$X = \frac{1}{M} \iint x\sigma\, dx\, dy = \frac{1}{M} \int_0^b \left(\int_0^a x\sigma_0 \frac{xy}{ab} dx \right) dy = \frac{1}{M} \int_0^b \left(\frac{\sigma_0 y}{ab} \frac{x^3}{3} \bigg|_0^a \right) dy$$

$$= \frac{1}{M} \frac{\sigma_0}{ab} \int_0^b \frac{ya^3}{3} dy = \frac{1}{M} \frac{\sigma_0}{ab} \frac{a^3}{3} \frac{b^2}{2} = \frac{4}{\sigma_0 ab} \frac{\sigma_0 a^2 b}{6} = \frac{2}{3} a$$

Similarly, $Y = \frac{1}{M} \iint y\sigma\, dx\, dy = \frac{1}{M} \int_0^a \left(\int_0^b y\sigma_0 \frac{xy}{ab} dy \right) dx = \frac{1}{M} \int_0^a \left(\frac{\sigma_0 x}{ab} \frac{y^3}{3} \bigg|_0^b \right) dx$

$$= \frac{1}{M} \frac{\sigma_0}{ab} \int_0^a \frac{xb^3}{3} dx = \frac{1}{M} \frac{\sigma_0}{ab} \frac{b^3}{3} \frac{a^2}{2} = \frac{4}{\sigma_0 ab} \frac{\sigma_0 b^2 a}{6} = \frac{2}{3} b$$

$\Rightarrow Y = \frac{2}{3} b$ So center of mass is $\vec{R} = \frac{2}{3}(a\hat{i} + b\hat{j})$

Example: Find the centre of semicircle ring of mass M and radius R.

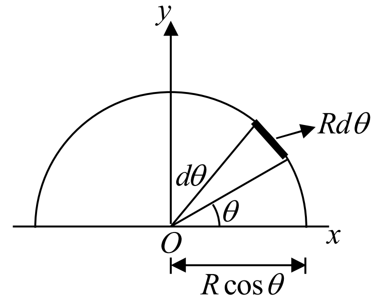
Solution: The length of semicircular ring is $= \pi R$

Mass per unit length of ring is $\frac{M}{\pi R}$ $\left(x = R\cos\theta, y = R\sin\theta \right)$

Length of the small arc subtending an angle $d\theta$ at the centre is $= Rd\theta$

Mass of the arc is $dm = Rd\theta \frac{M}{\pi R} = \frac{M}{\pi} d\theta$

Thus, the coordinate of the centre of mass



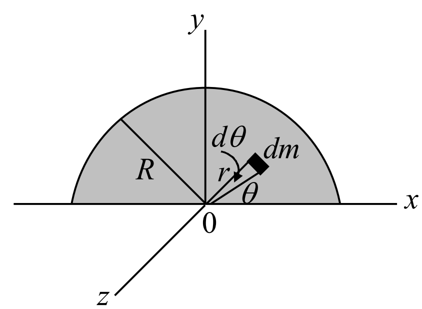
$$X_{C.M} = \frac{1}{M}\int xdm = \frac{1}{M}\int_0^{\pi}(R\cos\theta)d\theta = \frac{M}{\pi}d\theta = 0$$

$$Y_{C.M} = \frac{1}{M}\int ydm = \frac{1}{M}\frac{M}{\pi}\int_0^{\pi}(R\sin\theta)d\theta = \frac{M}{M}\frac{R}{\pi}[-\cos\theta]_0^{\pi}$$

$$= \frac{R}{\pi}(\cos\pi - \cos 0) = \frac{2R}{\pi} \qquad \therefore\ CM = \left(0, \frac{2R}{\pi}\right)$$

Example: Find the centre of mass of semicircular disc of mass M and radius R.

Solution: from the circular symmetry $x = r\cos\theta, y = r\sin\theta$



$$dm = \frac{2M}{\pi R^2}dx.dy = \frac{2M}{\pi R^2}rdr.d\theta$$

$$X_{C.M} = \frac{1}{M}\int xdm = \frac{2M}{M\pi R^2}\int_0^R\int_0^{\pi} r\cos\theta rdrd\theta = 0$$

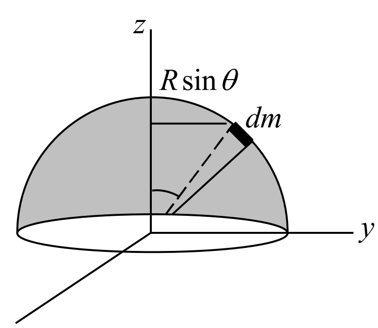
$$Y_{C.M} = \frac{1}{M}\int ydm = \frac{2M}{\pi R^2}\int_0^R\int_0^{\pi} r\sin\theta rdrd\theta$$

$$= \frac{2M}{M\pi R^2}\times\int_0^R r^2dr\int_0^{\pi}\sin\theta d\theta = \frac{2M}{M\pi R^2}\times\frac{R^3}{3}\times 2 = \frac{4R}{3\pi}$$

Example: Find the centre of mass of a hollow hemisphere of radius R and mass M

Solution: Consider a ring at angle θ from the base area of ring.

This problem can be solved in spherical symmetry



$$x = R\sin\theta\cos\phi, y = R\sin\theta\sin\phi, z = R\cos\theta$$

$$dm = \frac{M}{4\pi R^2}R^2\sin\theta d\theta d\phi$$

$$X_{C.M} = \int xdm = \frac{2MR^2}{2\pi R^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} R\sin\theta\cos\phi.\sin\theta d\theta d\phi$$

$$= \frac{MR^2}{4\pi R^2}\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta\int_0^{2\pi}\cos\phi d\phi = 0$$

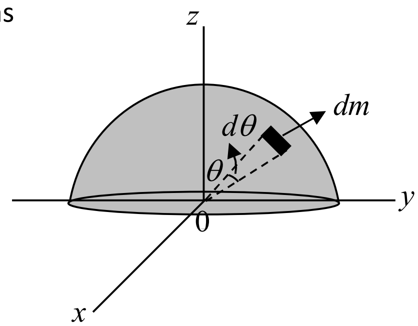
$$Y_{C.M} = \int ydm = \frac{MR^2}{2\pi R^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} R\sin\theta\sin\phi.\sin\theta d\theta d\phi = \frac{MR^2}{4\pi R^2}\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta\int_0^{2\pi}\sin\phi d\phi = 0$$

$$Z_{C.M} = \int z dm \quad = \frac{MR^2}{2\pi R^2}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} R\cos\theta.\sin\theta d\theta d\phi = \frac{MR^2}{\pi R^2}\frac{1}{2}\int_0^{\frac{\pi}{2}} R\sin 2\theta d\theta \int_0^{2\pi} d\phi = \frac{R}{2} = \frac{R}{2}$$

Example: Find the center of mass of a uniform solid hemisphere of radius R and mass M. From symmetry it is apparent that the center of mass lies on the z axis, as illustrated. Its height above the equatorial plane is

$$Z = \frac{1}{M}\int z dM \, .$$

Solution: We can solve this problem in spherical symmetry as



$x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$

$$dm = \frac{M}{\frac{4\pi R^3}{6}} dxdydz = \frac{6M}{4\pi R^3} r^2 dr \sin\theta d\theta d\phi$$

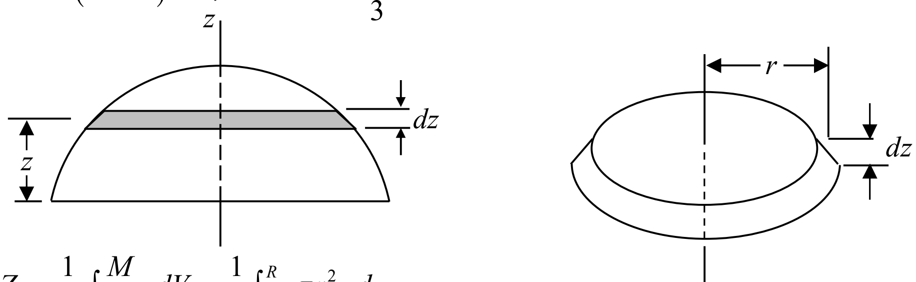
$$X_{C.M} = \frac{1}{M}\int x dm = 0, \quad Y_{C.M} = \frac{1}{M}\int y dm = 0$$

$$Z_{C.M} = \frac{1}{M}\times\frac{6M}{4\pi R^3}\times\int_0^R\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r\cos\theta . r^2 dr \sin\theta d\theta d\phi$$

$$\frac{1}{M}\times\frac{6M}{4\pi R^3}\times\int_0^R r^3 dr \int_0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta\int_0^{2\pi} d\phi = \frac{1}{M}\times\frac{6M}{4\pi R^3}\times\frac{R^4}{4}\times\frac{1}{2}\times 2\pi = \frac{3}{8}R$$

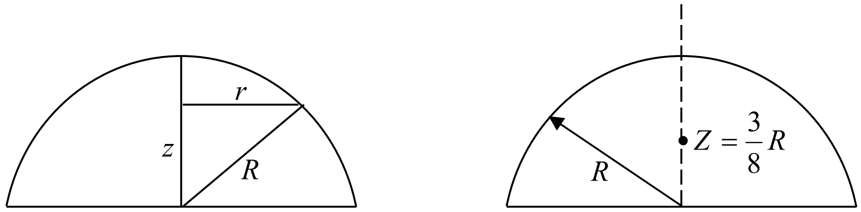
Second method

The integral is over three dimensions, but the symmetry of the situations lets us treat it as a one dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider the circular disk of radius r and thickness dz. Its volume is $dV = \pi r^2 dz$, and its mass is $dM = \rho dV = (M/V)dV$, where $V = \frac{2}{3}\pi R^3$.



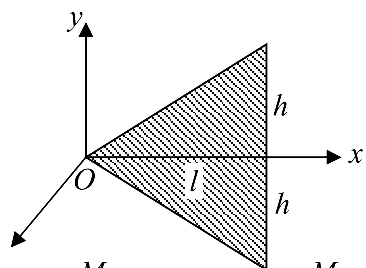
Hence, $Z = \frac{1}{M}\int\frac{M}{V} z dV = \frac{1}{V}\int_{z=0}^{R}\pi r^2 z\, dz$

To evaluate the integral, we need to find r in terms of z. Since, $r^2 = R^2 - z^2$, we have



$$Z = \frac{\pi}{V}\int_0^R z\left(R^2 - z^2\right)dz = \frac{\pi}{V}\left(\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\right)\Bigg|_0^R = \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right) = \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3} = \frac{3}{8}R.$$

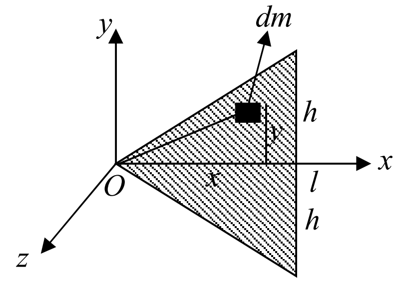
Example: Find the centre of mass triangular plate of Mass M. Where l and h are given parameter shown in figure.



Solution: $X_{CM} = \frac{\int x dm}{M}$ where $dm = \frac{M}{\frac{1}{2}2h.l}dx.dy \Rightarrow dm = \frac{M}{hl}dxdy$ and $r^2 = x^2 + y^2$

$$X_{CM} = \frac{M}{Mhl}\int_{x=0}^{x=l} xdx \int_{y=-\frac{h}{l}x}^{y=\frac{h}{l}x} dy = \frac{1}{hl}2.\frac{h}{l}\int_{x=0}^{x=l} x^2 dx = \frac{2l}{3}$$

$$Y_{CM} = \frac{\int ydm}{M} = X_{CM} = \frac{M}{Mhl}\int_{x=0}^{x=l} dx \int_{y=-\frac{h}{l}x}^{y=\frac{h}{l}x} ydy = 0$$

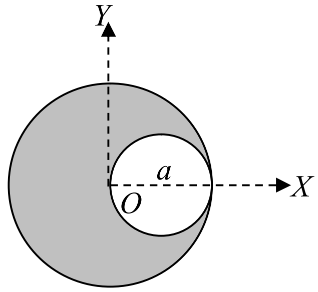


If some mass M_1 is removed from a rigid body of mass M, then the position of centre of mass of the remaining portion is obtained from the following formulae:

$$\vec{R}_{COM} = \frac{M\vec{r} - M_1\vec{r}_1}{M - M_1}$$

Here, $M, \vec{r}$ the values mass and center of mass of the whole mass while $M_1, \vec{r}_1$ are the values for the mass which has been removed.

Example: Find the position of centre of mass of the uniform lamina shown in figure.



Solution: Here, $A_1 = \text{area of complete circle} = \pi a^2$ the mass of disc is M

$(x_1, y_1) = \text{coordinates of centre of mass of large circle} = (0,0)$

$A_2 = \text{area of small circle} = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

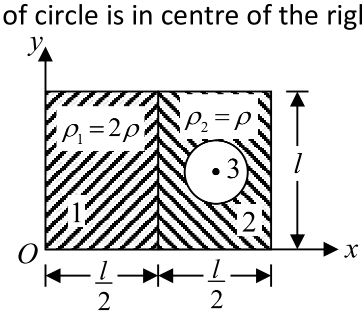
mass of removed circle is $M_1 = \frac{M}{\pi a^2} \times \frac{\pi a^2}{4} = \frac{M}{4}$

and $(x_2, y_2) = \text{coordinates of centre of mass of large circle} = \left(\frac{a}{2}, 0\right)$

$$X_{cm} = \frac{0.M - \frac{a}{2}.M_1}{M - M_1} = \frac{-\frac{a}{2}.\frac{M}{4}}{M - \frac{M}{4}} = -\frac{a}{6}, X_Y = \frac{0.M - 0M_1}{M - M_1} = \frac{0}{M - \frac{M}{4}} = 0$$

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$

Example: Find the position of centre of mass of the remaining plate when a circle is cut from right half radius of circle is $\frac{l}{8}$. Centre of circle is in centre of the right part of the plate.



Solution: Area $\quad \bar{x} \quad \bar{y}$

$A_1 \quad \frac{l^2}{4} \quad \frac{l}{4} \quad \frac{l}{2}$

A_2	$\frac{l^2}{4}$	$\frac{3l}{8}$	$\frac{l}{2}$
A_3	$\frac{\pi l^2}{64}$	$\frac{3l}{8}$	$\frac{l}{2}$

Taking origin at corner O

$$x_{COM}=\frac{\Sigma Ax}{\Sigma A}=\frac{A_1x_1+A_2x_2-A_3x_3}{A_1+A_2-A_3}$$

$$=\frac{2\rho\left(\frac{l^2}{4}\times\frac{l^2}{4}\right)+\rho\times\frac{l^2}{4}\times\frac{3l}{8}-\frac{\pi l^2}{64}\times\frac{3l}{8}\times\rho}{2\rho\frac{l^2}{4}+\frac{l^2}{4}\times\rho-\pi\frac{l^2}{64}\times\rho}=\frac{\frac{\rho l^3}{8}+\frac{3\rho l^3}{32}-\frac{3\pi\rho l^3}{512}}{\rho l^2\left(\frac{3}{4}-\frac{\pi}{64}\right)}=\frac{l\left(\frac{7}{32}-\frac{\pi\times 3}{512}\right)}{\frac{(48-\pi)}{64}}=0.28l$$

$$y_{COM}=\frac{\frac{l^2}{4}\times\frac{l}{2}\times 2\rho+\frac{l^2}{4}\times\frac{l}{2}\times\rho-\frac{\pi l^2}{64}\times\frac{l}{2}\times\rho}{2\rho\times\frac{l^2}{4}+\frac{l^2}{4}\rho-\frac{\pi l^2}{64}\times\rho}=\frac{l\left(\frac{1}{4}+\frac{1}{8}-\frac{\pi}{128}\right)}{\frac{1}{2}+\frac{1}{4}-\frac{\pi}{64}}=0.5l$$

Centre of mass is $(0.28l,\ 0.5l)$.