Chapter 1
Stability Analysis and Phase Diagram

2. Stability and Instability in One Dimension

Equilibrium Point: Any potential can be function of generalized coordinate, generalized velocity and time $V \equiv V(x, x, t)$. The equilibrium point is defined where total external force on the system is zero i.e. for any co-ordinate say x is said to be equilibrium point if $\frac{\partial V}{\partial x} = 0$ at $x = x_0$

Unstable Equilibrium Point: If x_0 is maxima or (local maxima) i.e. $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} < 0$, then it is said to be unstable equilibrium point. Unstable equilibrium point always behaves like repulsive point.

Stable Equilibrium Point: If x_0 is minima or (local minima) i.e. $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} > 0$, then it is said to be stable equilibrium point. Stable equilibrium point always behaves as an attractive point.

Example: If potential in one dimension is given by $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ then

(a) Find the point where potential is zero

(b) Find the equilibrium point.

(c) Find the stable and unstable equilibrium point

(d) Draw phase curve i.e. $V(x)$ vs x for given energy

Solution: (a) $V(x) = 0 \Rightarrow -\frac{x^2}{2} + \frac{x^4}{4} = 0 \Rightarrow x = 0, +\sqrt{2}, -\sqrt{2}$

(b) For equilibrium point $\frac{\partial V}{\partial x} = 0 \Rightarrow -x + x^3 = 0$. So there are three equilibrium points.

$x_1 = 0$, $x_2 = 1$, $x_3 = -1$

(c) For discussion of stability and instability, we must find, $\frac{\partial^2 V}{\partial x^2} = -1 + 3x^2$. For stable equilibrium $\frac{\partial^2 V}{\partial x^2} > 0$. At $x_2 = 1$ and $x_3 = -1$ the value of $\frac{\partial^2 V}{\partial x^2} = 2$ which is greater than 0. For unstable equilibrium point. $\frac{\partial^2 V}{\partial x^2} < 0$. At $x_1 = 0$, the value of $\frac{\partial^2 V}{\partial x^2} = -1$, which is less than 0, so it is unstable point.

(d) $V(x)$ vs x