Chapter 10

Identical Particles

3. Fermi Dirac Distribution

In quantum statistics, Fermi**–Dirac statistics** describes distribution of particles in a system comprising many identical particles that **obey the Pauli exclusion principle** (The **Pauli exclusion principle** is the quantum mechanical principle that no two identical fermions (particles with half-integer spin) may occupy the same quantum state simultaneously.)

Fermi–Dirac (F–D) statistics applies to identical particles with **half odd integer spin** in a system in thermal equilibrium. Additionally, the particles in this system are assumed to have negligible mutual interaction. This allows the many-particle system to be described in terms of single-particle energy states. The result is the F–D distribution of particles over these states and includes the condition that no two particles can occupy the same state, which has a considerable effect on the properties of the system. Since F–D statistics applies to particles with half-integer spin, these particles have come to be called fermions . It is most commonly applied to electrons, which are fermions with spin $\frac{1}{2}$

No. of ways W in which n_i indistinguishable particles to place in g_i level with the condition that only one particle or no particle can be placed in each level i.e. identical particles that **obey the Pauli exclusion principle.** (It is given $\sum_{i=1}^{i=l} n_i = N \quad \sum_{i=1}^{i=l} E_i n_i = U$.)

$$W = \sum_{i=1}^{l} \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i}$$

Fermi-Dirac distribution of the particles among various states is given by

$$n_i = \frac{g_i}{\exp(\alpha + \beta E) + 1}$$

So Fermi Dirac distribution $f(E) = \frac{n_i}{g_i} = \frac{1}{e^{(\alpha+\beta E)} + 1} = \frac{1}{Ae^{\beta E} + 1}$

where $\beta = \frac{1}{k_B T}$ and $A = e^{\alpha} = \frac{V}{N}\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$

when A.>> 1, Fermi-Dirac gas reduces to the Maxwell-Boltzmann gas. Fermi-Dirac gas is said to be weakly degenerate when A>1, degenerate when A<1 and strongly degenerate when A = 0.

Example: What is number of ways if two fermions have to adjust in energy state whose degeneracy is three.

Solution: $g_i = 3$, $n_i = 2$ $\quad W = \frac{\lfloor g_i}{\lfloor n_i \lfloor g_i - n_i} = 3$

Two indistinguishable particles is shown by A,A

Possible selection	1	2	3
1	A	A	0
2	0	A	A
3	A	0	A

Fermions at High Temperature

A gas of low density and high temperature is known as a weakly degenerate gas $(A>1)$. In a Fermi gas, total number N of particles and total energy U are

$g(E)dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}dE$ is density of state in three dimension with degeneracy $g = 2s+1$ where s is spin .

Total no of particle at temperature T is $N = \int_0^\infty f(E)g(E)dE = \int_0^\infty \frac{g(E)}{Ae^{\beta E}+1}dE$

Total energy at temperature T $U = \int_0^\infty E\ f(E)\ g(E)\,dE = \int_0^\infty \frac{Eg(E)}{Ae^{\beta E}+1}dE$

$$N = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{E^{1/2}}{Ae^{\beta E}+1}dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} F_{3/2}(A)$$

$$U = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{E^{3/2}}{Ae^{\beta E}+1}dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} F_{5/2}(A)$$

where $F_n(A) = \int_0^\infty \frac{E^{n-1}}{Ae^{\beta E}+1}dE$

Let $\beta E = u$, then $\beta dE = du$ and we have $F_n(A) = \frac{1}{\beta_n}\int_0^\infty \frac{u^{n-1}}{Ae^u+1}du = (k_BT)^n\,\Gamma(n)\,f_n(A)$

where $f_n(A) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{u^n-1}{Ae^u+1}du$ and Γ(n) is a gamma function.

$$N = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}(k_BT)^{3/2}\Gamma(3/2)f_{3/2}(A)$$

$$= gV\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2} f_{3/2}(A) = \frac{gV}{\lambda^3}f_{3/2}(A)$$

$$U = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}(kT)^{3/2}\Gamma(5/2)f_{5/2}(A)$$

$$= \frac{3}{2}kTgV\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2} f_{5/2}(A) = \frac{3}{2}k_BT\frac{gV}{\lambda^3}f_{5/2}(A)$$

where $f_n(A) = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{1}{A^r}\frac{1}{r^n}$

Very weakly degenerate gas

A gas with very large value of A i.e. at very high temperature T is known as very weakly degenerate gas. For that $f_n(A) = A^{-1}$.

$$N = \frac{gV}{\lambda^3} f_{3/2}(A) = \frac{gV}{\lambda 3}\frac{1}{A}$$

$$U = \frac{3}{2}k_B T\frac{gV}{\lambda^3} f_{5/2}(A) = \frac{3}{2}k_B T\frac{gV}{\lambda^3}\frac{1}{A}$$

Thus, we have $\frac{U}{N} = \frac{3}{2}k_B T$ $\qquad U = \frac{3}{2}Nk_B T$

and the specific heat at constant volume c_v is $c_V = \left[\frac{\partial U}{\partial T}\right]_V = \frac{3}{2}Nk_B$ so at very high temperature Fermi gas behave as ideal gas.

Weakly Degenerate Gas

For A>1, the gas is said to be weakly degenerate and equations and give

$$N = \frac{gV}{\lambda^3} f_{3/2}(A) \qquad U = \frac{3}{2}k_B T\frac{gV}{\lambda^3} f_{5/2}(A)$$

Therefore,

$$\frac{U}{N} = \frac{3}{2}k_B T\frac{f_{5/2}(A)}{f_{3/2}(A)} \qquad U = \frac{3}{2}Nk_B T\frac{f_{5/2}(A)}{f_{3/2}(A)}$$

$$= \frac{3}{2}Nk_B T\left[1 + \frac{1}{4\sqrt{2}}\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} + \left(\frac{1}{16} - \frac{4}{9\sqrt{3}}\right)\frac{N^2}{V^2}\left(\frac{h^2}{2\pi m k_B T}\right)^{\frac{3}{2}}\right]$$

$$c_V = \left[\frac{\partial U}{\partial T}\right]_V = \frac{3}{2}k_B N\left[1 - \frac{1}{8\sqrt{2}}\frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} - \left(\frac{1}{8} - \frac{4}{9\sqrt{3}}\right)\frac{N^2}{V^2}\left(\frac{h^2}{2\pi m k_B T}\right)^{3}\right]$$

Fermions at Low Temperature

Strongly degenerate Fermi gas $A < 1$

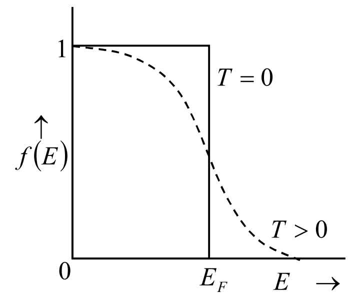
The Fermi-Dirac energy distribution is

$$f(E) = \frac{1}{\exp(E-\mu)/k_B T + 1}$$

where μ is the chemical potential which is a function of T, i.e; $\mu = \mu(T)$. the gas is strongly degenerate (A = 0) at T = 0. at T = 0, where $\mu = \mu(0) = E_F$. The limiting chemical potential is known as the Fermi energy E_F of the gas and the distribution function can be written as

$$f(E) = \frac{1}{e^{(E-E_F)/k_BT} + 1}$$

Strongly degenerate Fermi gas at $T = 0$



At T = 0, when E < E_F, we have $f(E) = \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1$

At T = 0, when E > E_F, we have $f(E) = \frac{1}{e^{\infty}+1} = \frac{1}{\infty+1} = 0$

Fermi function $f(E)$ versus E at T

The number of energy states in the energy range from E to $E + dE$

$g(E)dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2} dE$ Here, g is the spin degeneracy, $g = (2s+1)$, where s is the spin quantum number of a particle. The number of particles in the energy range from E to $E + dE$ at $T = 0$ is

$$n(E)dE = \begin{cases} f(E)g(E)dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} E^{1/2} dE & \text{for } E < E_F \\ 0 & \text{for } E > E_F \end{cases}$$

$$N = \int_0^{E_F} n(E)dE = 2g\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int_0^{E_F} E^{1/2}\, dE = \frac{4g\pi V}{3}\left(\frac{2m}{h^2}\right)^{3/2} E_F^{3/2}$$

Thus, the Fermi energy is $E_F(0) = \left(\frac{3N}{4\pi gV}\right)^{2/3} \frac{h^2}{2m}$ and the Fermi temperature T_F is defined as

$$T_F = \frac{E_F}{k_B}$$

The Fermi momentum p_F is given by

$$p_F = (2mE_F)^{1/2} = \left(\frac{3N}{4\pi gV}\right)^{1/3} h$$

Total energy of the gas at $T=0$ is

$$U=\int_0^\infty En(E)dE=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{E_F}E^{3/2}dE$$

$$=\frac{4g\pi V}{5}\left(\frac{2m}{h^2}\right)^{3/2}E_{E_F}^{5/2}=\frac{2g\pi V}{5mh^3}p_F^5$$

Thus, at $T=0$ we have $\dfrac{U}{N}=\dfrac{3}{5}E_F$

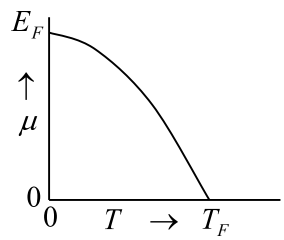


Fig: Variation of chemical potential μ with T.

Strongly degenerate case for $T>0$

- Chemical potential μ

$$\mu=E_F\left[1-\left(\frac{T}{T_F}\right)^2\frac{\pi^2}{12}.....\right],$$ where $T_F=E_F/k_B=\mu/k_B$ is the Fermi temperature.

Internal energy

$$U=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}F_{5/2}(A)=2g\pi V\left(\frac{2m}{h^2}\right)^{3/2}\frac{2}{5}\mu^{5/2}\left[1+\left(\frac{kT}{\mu}\right)^2\frac{5\pi^2}{8}+.....\right]$$

$$\frac{U}{N}=\frac{3}{5}\frac{\mu^{5/2}}{E_F^{3/2}}\left[1+\left(\frac{T}{T_F}\right)^2\frac{5\pi^2}{8}+.......\right]$$ Using the value of μ, we can approximate

$$\frac{U}{N}=\frac{3}{5}E_F\left[1-\left(\frac{T}{T_F}\right)^2\frac{\pi^2}{12}\right]^{5/2}\left[1+\left(\frac{T}{T_F}\right)^2\frac{5\pi^2}{8}\right]$$

$$=\frac{3}{5}E_F\left[1-\left(\frac{T}{T_F}\right)^2\frac{5\pi^2}{24}\right]\left[1+\left(\frac{T}{T_F}\right)^2\frac{5\pi^2}{8}\right]\quad=\frac{3}{5}E_F\left[1+\left(\frac{T}{T_F}\right)^2\frac{5\pi^2}{12}\right]$$

Thus, we have $U\approx\dfrac{3}{5}NE_F\left[1+\dfrac{5\pi^2}{12}\left(\dfrac{T}{T_F}\right)^2\right]$

- Hence, the specific heat at constant volume c_V is

$$C_V=\left[\frac{\partial U}{\partial T}\right]_V=\frac{3}{5}NE_F\frac{5\pi^2}{6}\frac{T}{T_F^2}=k_BN\frac{\pi^2}{2}\left(\frac{T}{T_F^2}\right)$$

- Pressure of the ideal Fermi gas is

$$P=\frac{2}{3}\frac{U}{V}=\frac{3}{5}\frac{NE_F}{V}\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

Example: Fermions of mass m are kept in two-dimensional box of area A at temperature $T=0$

(a) What is total number of particle if E_F is Fermi energy

(b) What is the energy of the system if E_F is Fermi energy

(c) Write expression of energy in term of E_F and N

For two dimensional systems density of state $g(E)dE = \pi A\left(\frac{2m}{h^2}\right)dE$ and distribution function

at temperature $T=0$ for is given by $f(E)=1$ if $E<E_F$

$=0$ if $E>E_F$

(a) $N=\int_0^{E_F} g(E)f(E)dE \quad \Rightarrow N=\frac{\pi 2mAE_F}{h^2}$

(b) $E=\int_0^{E_F} Eg(E)f(E)dE \quad E=\frac{\pi 2mA}{h^2}\int_0^{E_F} EdE, \quad E=\frac{\pi mAE_F^2}{h^2}$

(c) $E=\frac{NE_F}{2}$

Example: (a) if Fermi gas is at temperature $T>0$ what will $f\left(E_F\right)$

(b) At $E=E_F+x$, find the fraction of occupied levels

(c) At $E=E_F-x$, find fraction of unoccupied levels.

Solution: (a) It is also interesting to note that at $T>0$, when $E=E_F$ we have

$$f\left(E\right)=\frac{1}{e^{E-E_F/kT}+1}=\frac{1}{e^{0/kT}+1}=\frac{1}{1+1}=\frac{1}{2}$$

(b) At $T>0$, fraction of levels above E_F are occupied and a fraction of levels below E_F are vacant.

The fraction of occupied levels at the energy E is

$$f\left(E\right)=\frac{1}{e^{(E-E_F)/kT}+1}$$

At $E=E_F+x$, the fraction of occupied levels is $f\left(E_F+x\right)=\frac{1}{e^{x/kT}+1}$

(c) The fraction of unoccupied levels at the energy E is

$$1-f\left(E\right)=1-\frac{1}{e^{(E-E_F)/kT}+1}$$

At $E=E_F-x$, the fraction of unoccupied levels is $1-f\left(E_F-x\right)=\frac{1}{1+e^{x/kT}}$