Chapter 5

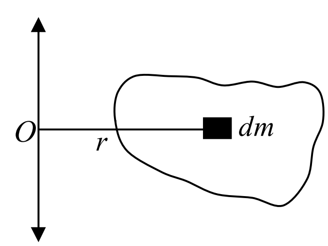
Centre of Mass and Moment of Inertia

3. Moment of Inertia

Newton's law of motion for linear motion can be written as $\sum_i \vec{F}_i = m\vec{a}$ where $\vec{F}_i$ is the external force on particle with mass m the effect of force on mass can be calculated as acceleration $\vec{a}$.

The mass m is also identify of measurement of Inertia.

Similarly for the Rotational motion of rigid body about any axis passing through O the equation of motion can be written as torque equation as $\sum_i \vec{\tau}_i = I_o\vec{\alpha}$ where $\vec{\tau}_i$ is external torque about axis passing through O and $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ is angular acceleration of Rigid body . I_O is Identified as Moment of Inertia of rigid body about axis passing through O



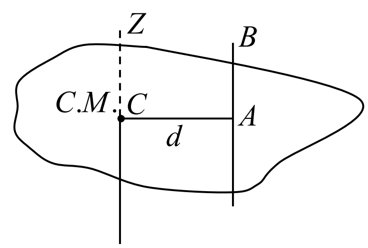
So moment of inertia about axis passing through axis O due to small elemental mass dm is given by $I_O = \int r^2 dm$ where r is perpendicular distance between elemental mass dm and axis.

For example if elemental mass dm is confined in x, y plane then moment of inertia about Y axis is $I_y = \int x^2 dm$.

To solve moment of inertia effectively we use two basic theorem.

1. Theorem of Parallel Axes

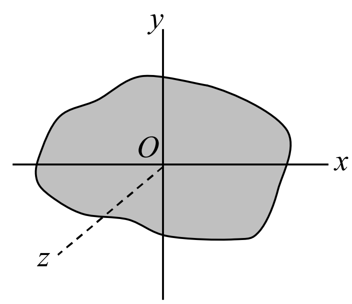
Suppose we have to obtain the moment of inertia of a body about a given line AB. Let C be the centre of mass of the body and Axis passing through center of mass is CZ be the line parallel to AB through C. Let I and I_C be the moment of inertia of the body about AB and CZ respectively. Then, $I = I_C + Md^2$, where d is the perpendicular distance between lines AB and CZ and M is mass of the body.



2. Theorem of Perpendicular Axes

The perpendicular axis theorem is used only when mass is distributed in plane. If mass is distributed in $x-y$ plane and I_X, I_Y and I_Z is principle moment of inertia about x, y, z axis respectively, then according to perpendicular axis theorem

$$I_Z = I_X + I_Y$$

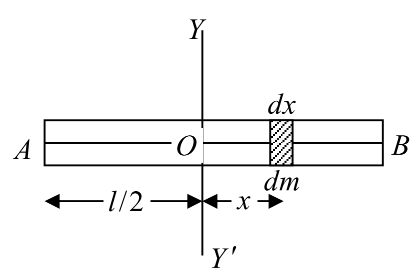


Moment of Inertia of Particular Cases

A. Uniform Rod

(i) Moment of inertia About an axis through its centre and perpendicular to its length.

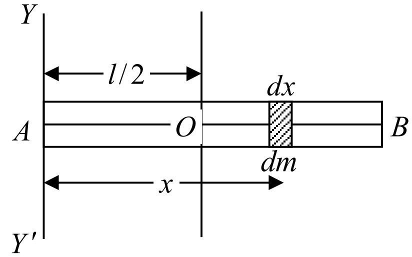
Let AB be a thin uniform rod, of length l and mass M, free to rotate about an axis YOY' passing through its centre O and perpendicular to its length, as shown in figure. Since the rod is uniform, its mass per unit length is clearly M/l. Considering a small element of the rod, of length dx at a distance x from the axis through O, we have mass of the element $dm = (M/l) \cdot dx$ and therefore, its moment of inertia about the axis (YOY') through



$$I_{CM} = I_y^{y'} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{M}{l} dx \, . \, 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{2M}{l} \cdot \frac{l^3}{24} = \frac{Ml^2}{12}$$

(ii) Moment of inertia about an axis through one end of the rod and perpendicular to its length.

Proceeding as in case (i) above, we obtain the moment of inertia of the rod about the axis, now passing through one end A of the rod, by integrating the expression for the moment of inertia of the element dx of the rod, between the limits $x = 0$ at A and $x = l$ at B, i.e.,



$$I = \int_0^l \frac{M}{l} x^2 dx = \frac{M}{l} \cdot \frac{l^3}{3} = \frac{Ml^2}{3}$$

Alternatively, we could obtain the same result by an application of the principle of parallel axes, according to which

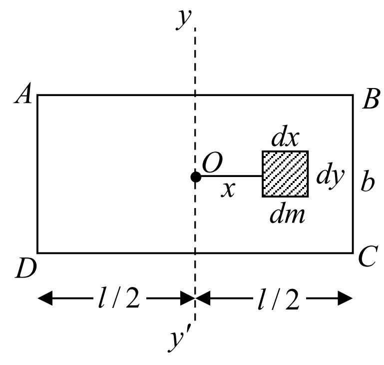
Moment of inertia of the rod about the axis $YAY' =$ Its moment of inertia about a parallel axis through O + (mass of the rod $\times$ square of the distance between the two axes).

So that, $I = I_{CM} + Md^2$, $I_{C.M} = \frac{ML^2}{12}, d = \frac{l}{2}$

$$\frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

B. Rectangular Lamina (or Bar)

(i) Moment of inertia About an axis through its centre and parallel to one side: Let $ABCD$ be a rectangular lamina, of length l, breadth b and mass M and let YOY' be the axis through its centre O and parallel to the side AD or BC about which its moment of inertia is to be determined.



Consider an element, or a small rectangular strip of the lamina, parallel to, and at a distance x from the axis. The area of this strip or element $dA = dx \times b$. And, since the mass per unit area of the lamina $= \frac{M}{(l \times b)}$, we have

mass of the strip or element $dm = \frac{M}{l \times b} \times dx \times b = \frac{M}{l} dx$.

And therefore, M.I. of the element about the axis $YOY' = I_Y^{Y'} = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{M}{l} dx$

The moment of inertia (I) of the whole rectangular lamina is then given by twice the integral of the above expression between the limits $x = 0$ and $x = \frac{l}{2}$.

i.e., $I_Y^{Y'} = I_{CM} = 2\frac{M}{l}\int_0^l x^2 dm = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}$

(ii) Moment of Inertia about side BC: From principle of parallel axes, $I = I_{C.M} + md^2$

$I_{C.M} = \frac{Ml^2}{12}$, $d = \frac{l}{2}$

So, moment of inertia about axis which is passing through BC $I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$.

This is again the same case as that of the M.I. of a rod about an axis passing through one of its ends and perpendicular to its length [case I (ii)], for, as pointed out above, if b be small, the rectangular lamina too reduces to a thin rod of length l.

(iii) Moment of inertia about an axis passing through its centre and perpendicular to its plane: This may be easily obtained by an application of the principle of perpendicular axes theorem to

$$I_Z = I_X + I_Y \quad I_Y = \frac{Ml^2}{12}$$

Similarly, $$I_X = \frac{Mb^2}{12}$$

Moment of inertia about axis passing through O and perpendicular to plane is

$$I_{CM} = I_Z = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M\left(l^2 + b^2\right)}{12}$$

(iv) Moment of inertia of Lamina about an axis passing through the mid-point of one side BC and perpendicular to its plane: In this case the axis passes through the mid-point of side AB or BC, say, and perpendicular to the plane of the lamina, so that it is parallel to the axis through O (the c.m. of the lamina) in case (iii).

In accordance with the principle of parallel axes, therefore, the M. I. of the lamina about this axis is given by $I = I_{C.M} + Md^2 \quad I_{CM} = \frac{M\left(l^2 + b^2\right)}{12} \quad d = \frac{l}{2}$

$$I = M\frac{\left(l^2+b^2\right)}{12} + M\left(\frac{l}{2}\right)^2 = M\left(\frac{l^2+b^2}{12}+\frac{l^2}{4}\right) = M\left(\frac{l^2}{3}+\frac{b^2}{12}\right)$$

And, if the axis passes through the mid-point of AB or DC, we, similarly have

$$I = I_{C.M} + Md^2 \quad I_{CM} = \frac{M\left(l^2+b^2\right)}{12} \quad d = \frac{b}{2}$$

$$I = M\left(\frac{l^2+b^2}{12}\right) + M\left(\frac{b}{2}\right)^2 = M\left(\frac{l^2+b^2}{12}+\frac{b^2}{4}\right) = M\left(\frac{l^2}{12}+\frac{b^2}{3}\right)$$

(v) About an axis passing through one of its corners D and perpendicular to its plane: Let the axis passes through the corner D of the lamina. Since it is perpendicular to the plane of the lamina, it is parallel to the axis through its centre of mass O in case (iii). Again, therefore, by the principle of parallel axes, we have moment of inertia of the rectangular lamina about this axis through D given by

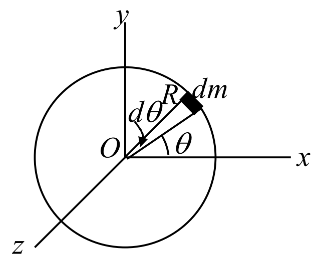
$$I = I_{CM} + Md^2, \quad I_{CM} = M\left(\frac{l^2+b^2}{12}\right)$$

where d is the distance between the two axes. Clearly, $d^2 = (l/2)^2 + (b/2)^2 = \left(l^2+b^2\right)/4$.

So that, $I = M\left(\frac{l^2+b^2}{12}\right) + M\left(\frac{l^2+b^2}{4}\right) = M\left(\frac{l^2+b^2+3l^2+3b^2}{12}\right) = M\left(\frac{l^2+b^2}{3}\right)$

C. Circular Ring

(i) Moment of inertia about an axis through its centre and perpendicular to its plane. Let the radius of the hoop or the thin circular ring be R and its mass M.



Consider a particle of mass m of the hoop or the ring. Clearly, its M.I. about an axis through the centre O of the hoop or the ring and perpendicular to its plane $I_Z = \int dmR^2$ where

$$dm = \frac{M}{2\pi R} R d\theta$$

$$I_Z = \int dmR^2 = \frac{M}{2\pi R} R^2 \int_0^{2\pi} Rd\theta = MR^2$$

(ii) Moment of inertia about axis, which is passing through its diameter: Obviously, due to symmetry, the M.I. of the hoop or the ring will be the same about one diameter as about another. Thus, if I be its M.I. about the diameter XOX' (Figure), it will also be I about the diameter YOY' perpendicular to XOX'.

By the principle of perpendicular axes theorem, $I_Z = I_X + I_Y$ From symmetry $I_X = I_Y$ and

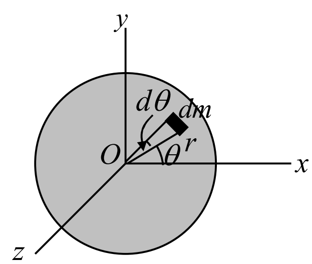
$I_Z = MR^2$ $\quad I_X + I_X = MR^2$ or $2I_X = MR^2$ hence, $I_X = \frac{MR^2}{2}$.

D. Circular lamina or disc

(i) Moment of inertia about an axis through its centre and perpendicular to its plane:

Let M be the mass of the disc and R, its radius, so that its mass per unit area is equal to $\frac{M}{\pi R^2}$.

Considering elemental mass of the disc dm, distance r from the axis passing through O and perpendicular to the plane of the disc, we have



$$dm = \left(\frac{M}{\pi R^2}\right) \times rdrd\theta$$

$$I_Z = \int r^2 dm = \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta = \frac{MR^2}{2}$$

(ii) Moment of inertia about axis, which is passing through its diameter: Here, again, due to symmetry, the M.I. of the disc, about one diameter is the same as about another. So that, if I be the M.I. of the disc about each of the perpendicular diameters XOX' and YOY', (Figure), By the principle of perpendicular axes theorem, $I_Z = I_X + I_Y$ From symmetry $I_X = I_Y$ and

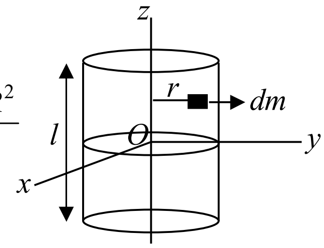
$I_Z = MR^2$ $\quad I_X + I_X = \frac{MR^2}{2}$ or.

i.e. $2I_X = \frac{MR^2}{2}$, hence, $I_X = \frac{MR^2}{4}$.

E. Solid Cylinder

(i) Moment of inertia of cylinder of radius R and uniform mass M about its own axis of cylindrical symmetry:

Using cylindrical symmetry, $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.



$$I_Z = \int r^2 dm \quad dm = \frac{M}{\pi R^2 l} r dr d\theta dz \Rightarrow I_Z = \frac{M}{\pi R^2 l}\int_0^R r^3 dr \int_0^{2\pi} d\theta \int_{-\frac{l}{2}}^{\frac{l}{2}} dz = \frac{MR^2}{2}$$

Other method:

A solid cylinder is just a thick circular disc or a number of thin circular disc (all of the same radius) piled up one over the other, so that its axis of cylindrical symmetry is the same as the axis passing through the centre of the thick disc (or the pile of thin discs) and perpendicular to its plane.

$\therefore$ M.I. of the thick disc (or the pile of thin discs) of the same mass and radius about the axis through its centre and perpendicular to its plane.

or $I = \frac{MR^2}{2}$

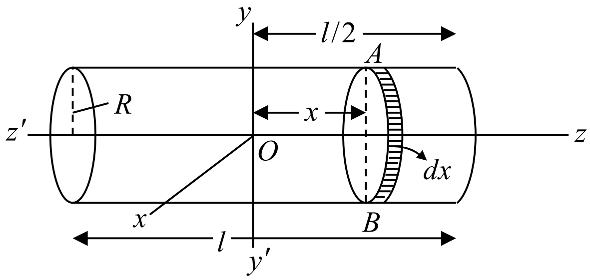
(ii) Moment of inertia About the axis through its centre and perpendicular to its axis of cylindrical symmetry: If R be the radius, l the length and M the mass of the solid cylinder, supposed to be uniform and of a homogeneous composition, we have its mass per unit length equal to $\frac{M}{l}$.

Now, imagining the cylinder to be made up of a number of discs each of radius R, placed adjacent to each other, and considering one such disc of thickness dx and at a distance x from the centre O of the cylinder, (figure), we have

Mass of the disc = $\left(\frac{M}{l}\right)dx$ and radius = R

And $\therefore$ M.I. of the disc about its diameter $AB = \frac{M}{l}dx \cdot \frac{R^2}{4}$ and its M.I. about the parallel axis YOY', passing through the centre O of the cylinder and perpendicular to its axis of cylindrical symmetry (or its length), in accordance with the principle of parallel axes,

$$= \frac{M}{l}dx\frac{R^2}{4} + \frac{M}{l}dx \cdot x^2$$



Hence, M.I. of the whole cylinder about this axis, i.e. $I =$ twice the integral of the above expression between the limits $x = 0$ and $x = \frac{l}{2}$,

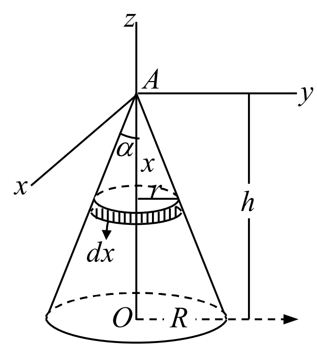
i.e., $$I = 2\int_0^{l/2}\left(\frac{M}{l}\cdot\frac{R^2}{4}dx + \frac{M}{l}x^2 dx\right) = \frac{2M}{l}\int_0^{l/2}\left(\frac{R^2}{4}dx + x^2 dx\right) = \frac{2M}{l}\left[\frac{R^2 x}{4} + \frac{x^3}{3}\right]_0^{l/2}$$

or $$I = \frac{2M}{l}\left[\frac{R^2}{4}\cdot\frac{l}{2} + \frac{l^3}{8\times 3}\right] = \frac{2M}{l}\left(\frac{R^2 l}{8} + \frac{l^3}{24}\right) = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)$$

F. Solid cone of mass M and its vertical height h and its base radius R (figure below).

(i) Moment of Inertia about its vertical axis.

Clearly, volume of the cone $= \frac{1}{3}\pi R^2 h$ and if ρ be the density of its material its mass $M = \frac{1}{3}\pi R^2 h\rho$, whence, $\rho = \frac{3M}{\pi R^2 h}$.



Now, the cone may be imagined to consist of a number of discs of progressively decreasing radii, from R to 0 piled up one over the other.

Considering one such disc of thickness dx and at a distance x from the vertex A of the cone, we have,

Radius of the disc, $r = x\tan\alpha$, where α is the semi-vertical angle of the cone, and therefore its volume $= \pi r^2 dx = \pi x^2 \tan^2\alpha dx$ and its mass $= \pi x^2 \tan^2\alpha\rho dx$.

Hence, M.I. of the disc about the vertical axis AO of the cone (i.e. an axis passing through its centre and perpendicular to its plane) = mass $\times\frac{(\text{radius})^2}{2}$

$$= \pi x^2\rho\tan^2\alpha\, dx\cdot\frac{r^2}{2} = \frac{\pi x^2\rho\tan^2\alpha\, dx\cdot x^2\tan^2\alpha}{2} = \left(\frac{\pi\rho\tan^4\alpha}{2}\right)x^4 dx$$

And since, M.I. of the entire cone about its vertical axis AO is given by

$$I = \int_0^h \frac{\pi\rho\tan^4\alpha}{2}x^4 dx = \frac{\pi\rho\tan^4\alpha}{2}\int_0^h x^4 dx = \frac{\pi\rho\tan^4\alpha}{2}\left[\frac{x^5}{5}\right]_0^h = \frac{\pi\rho\tan^4\alpha}{2}\cdot\frac{h^5}{5} = \frac{\pi\rho R^4}{2h^4}\cdot\frac{h^5}{5},$$

substituting R/h for $\tan\alpha$ or, substituting the value of ρ obtained above, we have

$$I = \frac{\pi\cdot 3M}{\pi R^2 h}\cdot\frac{R^4}{2h^4}\cdot\frac{h^5}{5} = \frac{3MR^2}{10}$$

(ii) Moment of inertia about an axis passing through the vertex and parallel to its base:

Again, considering the disc at a distance x from the vertex of the cone, we have its M.I. about its

diameter = $\frac{\text{mass}\times(\text{radius})^2}{4} = \pi x^2 \rho \tan^2\alpha dx\cdot\frac{r^2}{4} = \frac{\pi x^2\tan^2\alpha\rho dx\cdot x^2\tan^2\alpha}{4} = \left(\frac{\pi\rho\tan^4\alpha}{4}\right)x^4 dx$

$\therefore$ Its M.I. about the parallel axis XX', parallel to its base is given by

$$= \left(\frac{\pi\rho\tan^4\alpha}{4}\right)x^4 dx + \pi x^2\tan^2\alpha\rho x^2 dx = \left(\frac{\pi\rho\tan^4\alpha}{4}\right)x^4 dx + \pi\rho\tan^2\alpha x^4 dx$$

Hence M.I. of the entire cone about the axis XX', parallel to the base is given by

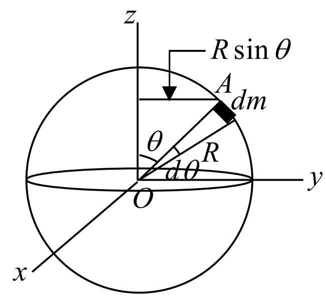
$$I = \int_0^h\left\{\frac{\pi\rho\tan^4\alpha}{4}x^4 dx + \pi\rho\tan^2\alpha x^4\right\}dx = \frac{\pi\rho\tan^4\alpha}{4}\int_0^h x^4 dx + \pi\rho\tan^2\alpha\int_0^h x^4 dx$$

$$= \frac{\pi\rho}{4}\cdot\frac{R^4}{h^4}\left[\frac{x^5}{5}\right]_0^h + \pi\rho\cdot\frac{R^2}{h^2}\left[\frac{x^5}{5}\right]_0^h = \frac{\pi\rho R^4}{4h^4}\cdot\frac{h^5}{5} + \frac{\pi\rho R^2}{h^2}\cdot\frac{h^5}{5} \qquad \left[\because \tan\alpha = \frac{R}{h}\right]$$

or, substituting the value of ρ, we have M.I. of the cone about the axis XX', i.e.

$$I = \frac{\pi\cdot 3M}{4\pi R^2 h}\cdot\frac{R^4}{h^4}\cdot\frac{h^5}{5} + \frac{\pi\cdot 3M}{4\pi R^2 h}\cdot\frac{R^2}{h^2}\cdot\frac{h^5}{5} = \frac{3MR^2}{20} + \frac{3Mh^2}{5}$$

G. Moment of inertia of a uniform hollow sphere about a diameter



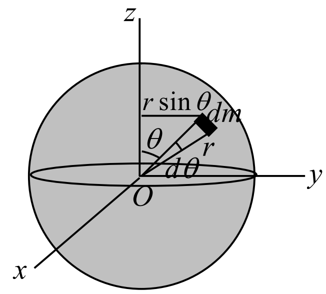
Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis. The mass is spread over the surface of the sphere and the inside is hollow.

$I_Z = \int r^2 dm$ where $dm = \frac{M}{4\pi R^2}R^2\sin\theta d\theta d\phi$ is elemental mass and

$r = R\sin\theta$, which is perpendicular distance from axis

$$I_Z = \frac{M}{4\pi R^2}\int_0^{\pi} R^2\sin^2\theta . R^2\sin\theta d\theta\int_0^{2\pi} d\phi = \frac{MR^4}{4\pi R^2}.\frac{4}{3}.2\pi = \frac{2MR^2}{3}$$

H. Moment of inertia of a uniform solid sphere about a diameter



Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis. The mass is spread over the surface of the sphere and the inside is hollow.

$I_Z = \int r^2 dm$ where $dm = \dfrac{M}{\frac{4\pi}{3}R^3} r^2 dr \sin\theta d\theta d\phi$ is elemental mass

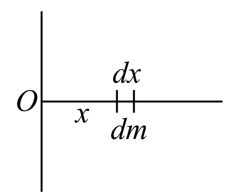
and $r = R\sin\theta$ which is perpendicular distance from axis

$$I_Z = \frac{3M}{4\pi R^3}\int_0^R r^4 dr \int_0^\pi \sin^2\theta . \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{3MR^5}{4\pi R^5}.\frac{4}{3}.2\pi = \frac{2MR^2}{5}$$

Radius of Gyration: The radius of Gyration K of body about axis is the effective distance from this axis where the whole mass can be assumed to be concentrated so that moment of inertia remain the same.

$$I = MK^2 \text{ so } K = \sqrt{\frac{I}{M}}$$

For example radius of gyration of disc about an axis perpendicular to its plane and passing through its center of mass is $K = \sqrt{\dfrac{I}{M}} = \sqrt{\dfrac{1MR^2}{2M}} = \dfrac{R}{\sqrt{2}}$

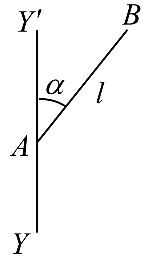


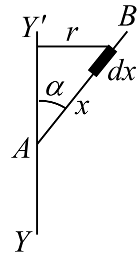
Example: If linear mass density of Rod is $\lambda = \dfrac{\lambda_0 x}{l}$ where l is length of Rod and x is distance measured from origin as shown in figure find the moment of inertia about axis passing through origin .

Solution: $\dfrac{dm}{dx} = \lambda \Rightarrow dm = \lambda dx \Rightarrow dm = \dfrac{\lambda_0 x}{l}dx$

$$I_Z = \int_0^l x^2 dm = \frac{\lambda_0}{l}\int_0^l x^2 .xdx = \frac{\lambda_0}{l}\int_0^l x^3 dx = \frac{\lambda_0 l^3}{4}$$

Example: Find the moment of inertia of rod AB of mass M and length l about an axis YY' are shown in figure.





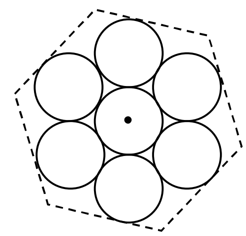
Solution: Elemental mass of the rod $dm = \frac{M}{l}dx$

Perpendicular distance of this elemental mass about YY' is $r = x\sin\alpha$

Moment of Inertia of this small element of the rod $I_Y^{Y'} = \int r^2 dm = \frac{M}{l}\sin^2\alpha x^2 dx$

$$I_Y^{Y'} = \frac{M}{l}\sin^2\alpha \int_{x=0}^{x=l} x^2 dx = \frac{M}{l}\sin^2\alpha \times \frac{l^3}{3} = \frac{Ml^2}{3}\sin^2\alpha$$

Example: Seven uniform rings, each of mass M and radius R, are inscribed inside a regular hexagon as shown.

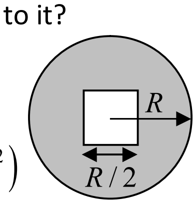


Find the moment of inertia of this system of seven rings, about an axis passing through the central ring and perpendicular to the plane of the disks

Solution: MI of central ring about axis $+6\times$ MI of surrounding Ring about axis

$$MR^2 + 6\times\left(MR^2 + M(2R)^2\right) = MR^2 + 6\times 5MR^2 = 31MR^2$$

Example: Consider a uniform thin circular disk of radius $2R$ and mass M. A concentric square of side R is cut out from the disk (see figure). What is the moment of inertia of the resultant disk about an axis passing through the centre of the disk and perpendicular to it?



Solution: Moment of Inertia of about center is $\frac{M}{2}(2R)^2 = 2MR^2$

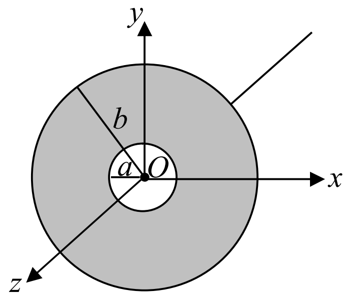
Moment of inertia of square of length a about center is $I_{square} = \frac{M_1}{12}\left(a^2 + a^2\right)$

put $a = R$ $I_{square} = \frac{M_1}{12}\left(R^2 + R^2\right) = \frac{M_1}{6}R^2$

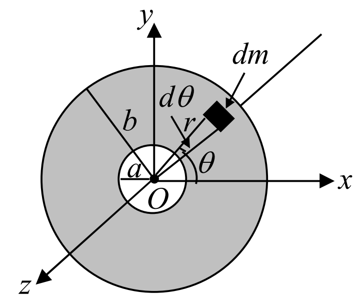
where $M_1 = \frac{M}{\pi(2R)^2}\times a\times a = \frac{M}{4\pi}$ put $a = R$, $M_1 = \frac{M}{4\pi}$ so $I_{sqare} = \frac{M}{24\pi}R^2$

$$I = I_{disc} - I_{square} = 2MR^2 - \frac{MR^2}{24\pi} = MR^2\left[2 - \frac{1}{24\pi}\right]$$

Example: Find the Moment of inertia of Annular Disc of mass M about axis passing through center and perpendicular to plane the inner radius of disc is a and outer radius is b



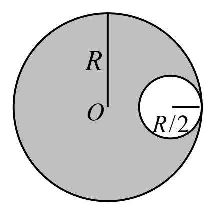
Solution: If b and a be the outer and inner radii of the disc having mass M, we have mass per unit area of the disc $\frac{M}{\pi\left(b^2-a^2\right)}$.



The elemental mass $dm = \frac{M.rdrd\theta}{\pi\left(b^2-a^2\right)}$

$$I_Z = \int r^2 dm = \frac{M}{\pi\left(b^2-a^2\right)}\int_a^b r^4 dr \int_0^{2\pi} d\theta \Rightarrow I_Z = \frac{M\left(a^2+b^2\right)}{2}$$

Example: Consider a uniform thin circular disk of radius R and mass M. A hole of radius $\frac{R}{2}$ is cut out from the disk (see figure). What is the moment of inertia of the resultant disk about an axis passing through the centre of the disk O and perpendicular to it?



Solution: Moment of inertia of disc $I_{OD} = \frac{MR^2}{2}$

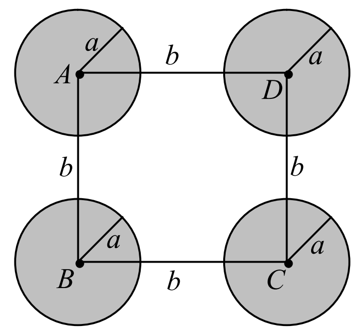
Moment of inertia of hole $I_{OH} = \frac{M_1R_1^2}{2} + M_1d^2$ where $R_1 = \frac{R}{2}$, $d = \frac{R}{2}$

And $\frac{M_1}{\pi R_1^2} = \frac{M}{\pi R^2} \Rightarrow M_1 = M \times \left(\frac{R_1}{R}\right)^2 = \frac{M}{4}$

$$I_{OH} = \frac{1}{2}\frac{M}{4}\times\left(\frac{R}{2}\right)^2 + \frac{M}{4}\left(\frac{R}{2}\right)^2 = \frac{3}{32}MR^2$$

$$I_O = I_{OD} - I_{OH} = \frac{MR^2}{2} - \frac{3}{32}MR^2 = \frac{13}{32}MR^2$$

Example: Four spheres of diameter $2a$ and mass M each are placed with their centres on the four corners of a square of side b.
(a) Find the moment of inertia about axis passing through center of square and perpendicular to plane of square
(b) Find the moment of inertia of the system about one side of the square taken as the axis



Solution: (a) Moment of inertia of each sphere about its diameter is $I_{CM} = \frac{2}{5}MR^2$

Moment of inertia of sphere about axis passing through center of square and perpendicular to plane of square $I_Z = I_{CM} + Md^2$, where $d = \sqrt{\frac{b^2}{4} + \frac{b^2}{4}} = \frac{b}{\sqrt{2}}$

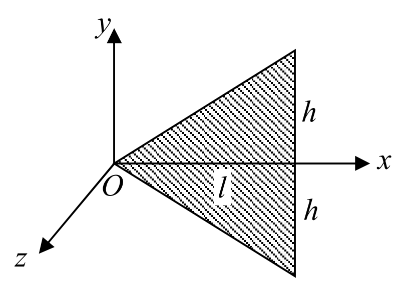
$$I_Z = \frac{2}{5}Ma^2 + \frac{Mb^2}{2}$$

The moment of Inertia of whole system is $I_{ZS} = 4 \times I_Z = \frac{8}{5}Ma^2 + \frac{4}{2}Mb^2 = \frac{8}{5}Ma^2 + 2Mb^2$

(b) The moment of inertia of the system about any side (say CD)

$I = M.I$ of A about $CD +$ moment of inertia of B about $CD + MI$ of C about $CD + MI$ of D about CD $= \left(\frac{2}{5}Ma^2 + Mb^2\right) + \left(\frac{2}{5}Ma^2 + Mb^2\right) + \frac{2}{5}Ma^2 + \frac{2}{5}Ma^2 = \frac{2}{5}M\left(4a^2 + 5b^2\right)$

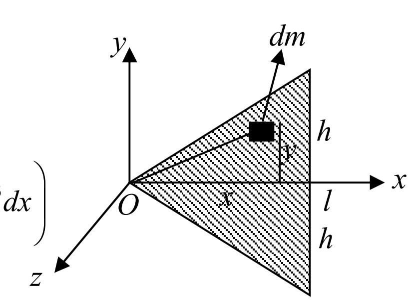
Example: Find the moment of Inertia about the axis passing through O and perpendicular to triangular plate of Mass M. Where l and h are given parameter shown in figure.



Solution: $I_Z = \int r^2 dm$ where

$dm = \frac{M}{\frac{1}{2}2h.l}dx.dy \Rightarrow dm = \frac{M}{hl}dxdy$ and $r^2 = x^2 + y^2$

$$I_z = \frac{M}{hl}\int_{x=0}^{x=l}\int_{y=-\frac{h}{l}x}^{y=\frac{h}{l}x}\left(x^2 + y^2\right)dxdy = 2\frac{M}{hl}\int_{x=0}^{l}\int_{0}^{y=\frac{h}{l}x}\left(x^2 + y^2\right)dxdy$$



$$2\frac{M}{hl}\left(\int_{x=0}^{l}\int_{0}^{y=\frac{h}{l}x}x^2 dxdy + \int_{x=0}^{l}\int_{0}^{y=\frac{h}{l}x}y^2 dxdy\right) = \frac{2M}{hl}\left(\frac{h}{l}\int_0^l x^3 dx + \int_0^l \frac{h^3}{3l^3}x^3 dx\right)$$

$$\frac{2M}{hl}\left(\frac{hl^4}{4} + \frac{h^3}{3l^3}\frac{l^4}{4}\right) = \left(\frac{Ml^2}{2} + \frac{Mh^2}{6}\right)$$