Chapter 1

# Stability Analysis and Phase Diagram

## 3. Small Oscillations

Let us assume the potential $V(x)$ has stable equilibrium point at $x = x_0$ then $\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0$ and $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} > 0$

The Taylor expansion of $V(x)$ about $x = x_0$ is given by

$$V(x) = V(x_0) + \left.\frac{\partial V}{\partial x}\right|_{x=x_0}(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0)^2 + \text{order}(x - x_0)^3 \ldots\ldots$$

If term $(x - x_0)^2$ is small then higher order terms can be neglected, then potential energy is equivalent to $V(x) = V(x_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0)^2$ because $\left.\frac{\partial V}{\partial x}\right|_{x=x_0} = 0$

So, force is equal to $F = -\frac{\partial V}{\partial x}$, $F = -\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x-x_0)$ . Hence, $F \propto -(x-x_0)$ and the motion is small oscillation and the angular frequency is given by $\omega = \sqrt{\frac{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}}{m}}$, where $m$ is mass of the particle. The term $k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}$ is identified as spring constant.

**Example:** If particle of mass $m$ interact with potential $ax^2 + \frac{b}{x^2}$, then what will be the frequency of oscillation? (Assume oscillation is small)?

**Solution:** $V(x) = ax^2 + \frac{b}{x^2}$

For equilibrium point $\frac{\partial V}{\partial x} = 0$

$$2ax - \frac{2b}{x^3} = 0 \Rightarrow ax^4 - b = 0 \Rightarrow x_0 = \pm\left(\frac{b}{a}\right)^{1/4}$$

$k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = 2a + \frac{2.3b}{x^4}$ . Now, put the value of $x_0 = \pm\left(\frac{b}{a}\right)^{1/4}$

$$k = 2a + \frac{6b \times a}{b} = 8a \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8a}{m}}$$