Chapter 4

Central Force and Kepler's System

3. Two Body Problem

Reduction of two body central force problem to the equivalent one body problem.

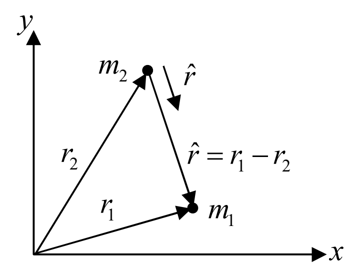


Figure 10

A system of two particles of mass m_1 and m_2 whose instantaneous position vectors of inertial frame with origin O are r_1 and r_2 respectively.

Vector m_2 relative to m_1 is $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

Central Force Motion as a One Body Problem

Consider an isolated system consisting of two particles interacting under a central force $f(r)$. The masses of the particles are m_1 and m_2 and their position vectors are r_1 and r_2. We have

$$r = r_1 - r_2 \Rightarrow r = |r| = |r_1 - r_2|$$

The equations of motion are

$$m_1\ddot{r}_1 = f(r)\hat{r} \qquad ....(1)$$

$$m_2\ddot{r}_2 = -f(r)\hat{r} \qquad ....(2)$$

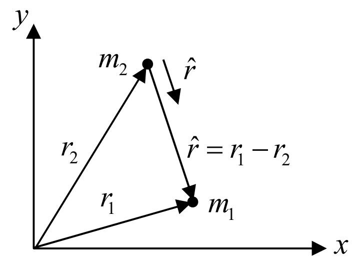


Figure 11

The force is attractive for $f(r) < 0$ and repulsive for $f(r) > 0$.Above equation are coupled Equations are coupled together by r; the behavior of r_1 and r_2 depends on $r = r_1 - r_2$.

To find the equation of motion for r we divide equation by m_1 (1) and equation by m_2 (2) and subtract.

This gives $\ddot{r}_1 - \ddot{r}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{r} \Rightarrow \left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{r}_1 - \ddot{r}_2) = f(r)\hat{r}$

Where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is identified as reduce mass and equation of motion is written in form of reduced mass as $\mu\ddot{r} = f(r)\hat{r}$ the system can be shown as given in figure.

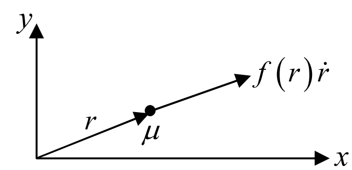


Figure 12

Equation is identical to the equation of the motion for a particle of mass μ acted on by a force $f(r)\hat{r}$; no trace of the two particle problem remains.

The two particle problem has been transformed to a one particle problem.

Transformation of two body on center of mass reference frame and calculation of energy

We shall show that the problem is easier to handle if we replace r_1 and r_2 by $r = r_1 - r_2$ and the center of mass vector $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$. The equation of motion for R is trivial since there are no external forces. The equation for r turns out to be like the equation of motion of a single particle and has a straight forward solution.

The equation of motion for R is $\ddot{R} = 0$, which has the simple solution $R = R_0 + vt$

The constant vectors R_0 and v depend on the choice of coordinate system and the initial conditions. If we are clever enough to take the origin at the center of mass, $R_0 = 0$ and $v = 0$.

$$r = r_1 - r_2 \quad R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

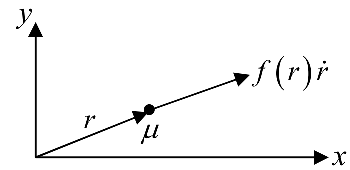


Figure 13

Solving for r_1 and r_2 gives

$$r_1 = R + \left(\frac{m_2}{m_1 + m_2}\right) r \text{ and } r_2 = R - \left(\frac{m_1}{m_1 + m_2}\right) r$$

For a simple trick lets fixed center of mass at origin ie for put $R = 0$

$$m_1 r_1 + m_2 r_2 = 0 \text{ and } \vec{r}_1 - \vec{r}_2 = \vec{r}$$

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \Rightarrow \dot{\vec{r}}_1 = \frac{m_2 \dot{\vec{r}}}{m_1 + m_2} \text{ and } \vec{r}_2 = -\frac{m_1 \vec{r}}{m_1 + m_2} \Rightarrow \dot{\vec{r}}_2 = -\frac{m_1 \dot{\vec{r}}}{m_1 + m_2}$$

Total energy of system is given by

$$E = \frac{1}{2} m_1 \left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2} m_2 \left|\dot{\vec{r}}_2\right|^2 + V\left(\vec{r}_1 - \vec{r}_2\right)$$

Put value of $\dot{\vec{r}}_1 = \frac{m_2 \dot{\vec{r}}}{m_1 + m_2}$ and $\dot{\vec{r}}_2 = -\frac{m_1 \dot{\vec{r}}}{m_1 + m_2}$

The energy of system is

$$E = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left|\dot{\vec{r}}\right|^2 + V(r) \Rightarrow E = \frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2 + V(r),$$

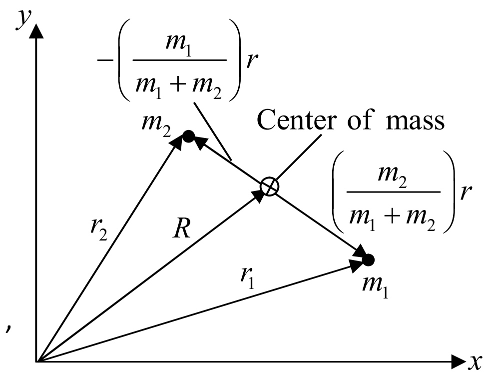


Figure 14

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is identified as reduced mass.

Hence $V(r)$ is central potential so motion is then motion is confided in the plane

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

Kepler's Law

Kepler discuss the orbital motion of the sun and Earth system under the potential, $V(r) = -\frac{k}{r}$ where $k = G m_s m_e$, it is given m_s and m_e is mass of Sun and Earth. Although Kepler discuss sun and earth system but method can be used for any system which is interacting with potential $V(r) = -\frac{k}{r}$

The reduce mass for sun and earth system is $\mu = \frac{m_e m_s}{m_e + m_s} \Rightarrow \frac{m_e}{1+\frac{m_e}{m_s}} = m_e \, , m_s >> m_e$

Let us assume mass of earth $m_e = m$

Kepler's First Law

Every planet (earth) moves in an elliptical orbit around the sun, the sun is being at one of the foci.

Where sun and earth interact each other with potential $V(r) = -\frac{k}{r}$ we solve equation of motion in center of mass reference frame with reduce mass $\mu = m_e = m$

Equation of Motion

Now, we discuss the case specially of elliptical orbit as Kepler discuss for planetary motion.

Total energy, $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{k}{r}$ where $V_{effective} = \frac{J^2}{2mr^2} - \frac{k}{r}$ with constant angular momentum J.

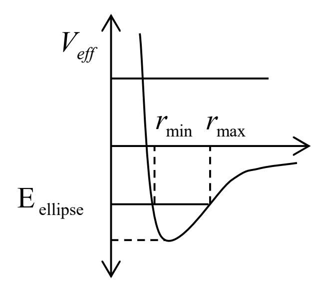


Figure 13

If one will plot $V_{effective}$ vs r, it is clear that for negative energy the orbit is elliptical which is shown in figure.

Earth is orbiting in elliptical path with sun as focus as shown in figure.

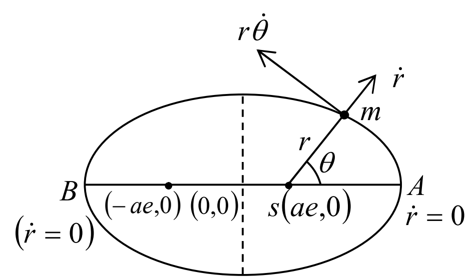


Figure 14

Let equation of this ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $b = a\sqrt{1-e^2}$

Minimum value of r is $(a - ae)$ and maximum value of r is $(a + ae)$, $r_{\max} + r_{\min} = 2a$.

From plot of effective potential, it is identified $r_{\max}$ and $r_{\min}$ is the turning point, so at these points radial velocity is zero.

$E=\frac{J^2}{2mr^2}-\frac{k}{r}\Rightarrow 2mEr^2+2mkr-J^2=0$ given equation is quadratic in term of r, for their root at $r_{\max}$ and $r_{\min}$. Using theory of quadratic equation sum of root, $r_{\max}+r_{\min}=-\frac{2mk}{2mE}\Rightarrow E=-\frac{k}{2a}$, which is negative.

Kepler's Second Law

Equal Area will swept in equal time or Areal velocity is constant.

$\frac{dA}{dt}=\frac{J}{2m}$ (which is derived earlier)

Kepler's Third Law:

The square of time period (T) of revolution in elliptical orbit is proportional to cube of semi major axis a i.e., $T^2\propto a^3$

$\frac{dA}{dt}=\frac{J}{2m}\Rightarrow\int dA=\frac{J}{2m}\int dt\ \Rightarrow\pi ab=\frac{J}{2m}T$ (πab is the area of ellipse)

$\pi a\cdot a\sqrt{1-e^2}=\frac{J}{2m}T$ it is given $e=\sqrt{1+\frac{2EJ^2}{mk^2}}$ and $E=-\frac{k}{2a}$

then, $e^2=1-\frac{2kJ^2}{2amk^2}\ \Rightarrow 1-e^2=\frac{2kJ^2}{2amk^2}=\frac{J^2}{amk}$

Now, $T^2=\frac{4m^2}{J^2}\pi^2a^2\cdot a^2(1-e^2)$ put value of $1-e^2=\frac{2kJ^2}{2amk^2}=\frac{J^2}{amk}$

$$\Rightarrow T^2=\frac{4m^2}{J^2}\pi^2a^4.\frac{J^2}{mak}=\frac{4\pi^2ma^3}{k}$$

If $k=Gm_sm$ then $\Rightarrow T^2=\frac{4\pi^2a^3}{Gm_s}$ where m_s is mass of the Sun.

Example: Given a classical model of tritium atom with nucleus of charge +1 and a single electron in a circular orbit of radius r_0, suddenly the nucleus emits a negatron and changes to charge +2 (the emitted negatron escapes rapidly and we can forget about it) the orbit suddenly has a new situation.

(a) Find the ratio of the electron's energy after to before the emission of the negatron

(b) Describe qualitatively the new orbit

(c) Find the distance of closest and the farthest approach for the new orbits in units of r_0

Solution: (a) As the negatron leaves the system rapidly, we can assume that its leaving has no effect on the position and kinetic energy of the orbiting electron.

From the force relation for the electron, $\frac{mv_0^2}{r_0} = \frac{e^2}{4\pi\varepsilon_0 r_0^2}$,

And we find its kinetic energy $\frac{mv_0^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r_0}$

And its total mechanical energy, $E_1 = \frac{mv_0^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{8\pi\varepsilon_0 r_0}$

Before the emission of the negatron. After the emission the kinetic energy of the electron is still $\frac{e^2}{8\pi\varepsilon_0 r_0}$, while its potential energy suddenly changes to $\frac{-2e^2}{4\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0}$

Thus after the emission the total mechanical energy of the orbiting electron is

$$E_2 = \frac{mv_0^2}{2} - \frac{2e^2}{4\pi\varepsilon_0 r_0} = \frac{-3e^2}{8\pi\varepsilon_0 r_0}, \text{ giving } \frac{E_2}{E_1} = 3.$$

In other words, the total energy of the orbiting electron after the emission is three times as large as that before the emission.

(b) As $E_2 = \frac{-3e^2}{8\pi\varepsilon_0 r_0}$, the condition for circular motion is no longer satisfied and the new orbit is an ellipse.

(c) Conservation of energy gives $\frac{-3e^2}{8\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0} + \frac{m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}{2}$

At positions where the orbiting electron is at the distance of closest or farthest approach to the atom, we have $\dot{r} = 0$,

for which $\frac{-3e^2}{8\pi\varepsilon_0 r_0} = \frac{mr^2\dot{\theta}^2}{2} - \frac{e^2}{2\pi\varepsilon_0 r} = \frac{J^2}{2mr^2} - \frac{e^2}{2\pi\varepsilon_0 r}$

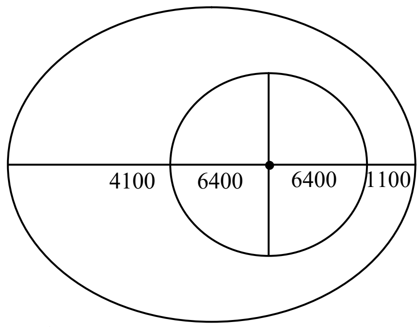
Then with, $J^2 = m^2 v_0^2 r_0^2 = \frac{me^2 r_0}{4\pi\varepsilon_0}$

The above becomes $3r^2 - 4r_0 r + r_0^2 = 0$ with solutions $r = \frac{r_0}{3}, \quad r = r_0$

Hence the distances of closest and farthest approach in the new orbit are respectively

$$r_{\min} = \frac{1}{3}, \quad r_{\max} = 1$$

Example: A satellite of mass $m = 2000\,kg$ is in elliptical orbit about earth. At perigee it has an altitude of 1,100 km and at apogee it has altitude 4,100 km. assume radius of the earth is $R_e = 6400\,km$. It is given $GmM_e = 8\times10^{17}\,Jm$



(a) What is major axis of the orbit?

(b) What is eccentricity of the orbit?

(c) What is angular momentum of the satellite?

(d) How much energy is needed to fix satellite in to orbit from surface of earth?

Solution: $r_{\max} = 4100 + 6400 = 10500\,km$

$r_{\min} = 1100 + 6400 = 7500\,km$

(a) $r_{\max} + r_{\min} = 2a \Rightarrow 18000 = 2a \Rightarrow a = 9000\ km$

(b) $e = \dfrac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} \Rightarrow e = \dfrac{10500 - 7500}{10500 + 7500} = \dfrac{3000}{18000} = \dfrac{1}{6} \Rightarrow e = \dfrac{1}{6}$

It is given $k = 8\times10^{17}\,J\cdot m$

(c) $E = -\dfrac{k}{2a} = -\dfrac{8\times10^{17}}{18000\times10^{3}}$, $E_f = -\dfrac{8}{18}\times10^{11}J = -4.5\times10^{10}J$

$$e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \Rightarrow \left(\frac{1}{6}\right)^2 = 1 + \frac{2EJ^2}{mk^2}$$

$$\frac{1}{36} - 1 = \frac{2EJ^2}{mk^2} \Rightarrow 140\times10^{26} = J^2 \Rightarrow J = \sqrt{140}\times10^{13} = 1.2\times10^{14}\,kgm/\sec^2$$

(d) When satellite is at surface of earth,

$R = 6400\ Km$

$$E_i = -\frac{GMm}{R} = \frac{-8\times10^{17}}{6400\times10^3} = \frac{-10^{17}}{800\times10^3} = \frac{-10^{12}}{8} = -12.5\times10^{10}$$

$$E_f = -\frac{GMm}{2a} = -4.5\times10^{10}\,J \Rightarrow \Delta E = E_f - E_i = 8\times10^{10}\,J$$

Example: For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.

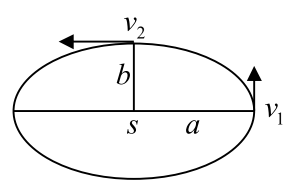
Solution: For Kepler's problem, $\frac{l}{r} = 1 + e\cos\theta$, for circular orbit $e = 0 \Rightarrow \frac{l}{r_c} = 1$

And for parabola $e = 1$, $\frac{l}{r_p} = 1 + \cos\theta$, r_p is minimum when $\cos\theta$ is maximum.

$\frac{l}{r_p} = 2$ and $\frac{l}{r_c} = 1 \Rightarrow \frac{r_p}{r_c} = \frac{1}{2}$

Example: A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b, respectively, the find total energy of the planet by assuming sun is at center of ellipse.

Solution: Assume Sun is at the centre of elliptical orbit.



Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$

Conservation of momentum $L = mv_1 a = mv_2 b$

$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{a} = GMm\frac{b}{a}\frac{1}{(b+a)} - \frac{GMm}{a}$$

$$= \frac{GMm}{a}\left(\frac{b}{(b+a)} - 1\right) = \frac{GMm}{a}\left(\frac{b-b-a}{(b+a)}\right) = -\frac{GMm}{(b+a)}$$