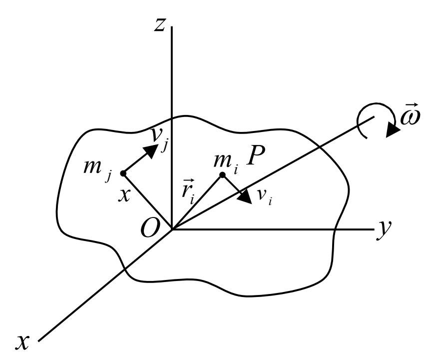
Chapter 5

Centre of Mass and Moment of Inertia

4. Moment of Inertia Tensor

Angular Momentum and Inertia Tensor

Let us consider the motion of a rigid body rotating about a fixed point o in the body as shown in figure.



At any instant of time, the body will rotate with velocity ω about that instant through point O.

A particle P of the body, having the position vector r_1 with respect to O has instantaneous velocity v_i relative to O, given by

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \text{, where } \vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Angular momentum of Point P about the point O is given by

$$\vec{J}_P = \vec{r}_i \times m_i \vec{v}_i \Rightarrow \vec{J}_P = \vec{r}_i \times m_i\left(\vec{\omega} \times \vec{r}_i\right)$$

$$\vec{J} = \sum_{i=1}^{N} \vec{r}_i \times m_i \vec{v}_i = \sum_{i}^{N} m_i \vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right) \Rightarrow \vec{J} = \sum_i m_i\left[r_i^2 \omega - \left(r_i \omega\right) r_i\right]$$

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$I_\alpha = \sum_\beta I_{\alpha\beta}\omega_\beta$, where $I_{\alpha\beta}$ is inertia tensor

and $I_{XX} = \sum_i m_i(y_i^2 + z_i^2)$, $I_{YY} = \sum_i m_i(x_i^2 + z_i^2)$, $I_{ZZ} = \sum_i m_i(x_i^2 + y_i^2)$ are known as moment of inertia and $I_{XY} = I_{YX} = -\sum_i m_i x_i y_i$, $I_{XZ} = I_{ZX} = -\sum_i m_i x_i z_i$, $I_{YZ} = I_{ZY} = -\sum_i m_i y_i z_i$ are known as product of inertia.

$$I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{N} m_i\left[\delta_{\alpha\beta} r_i^2 - x_{i\alpha} x_{i\beta}\right]$$

where $\alpha = 1,2,3$ and $\beta = 1,2,3$, so one can denote x, y, z by x_1, x_2, x_3 respectively.

For continuous system $I_{\alpha\beta}$ is reduce to

$\int dm\left[\delta_{\alpha\beta} x_i^2 - x_{i\alpha} x_{i\beta}\right]$ where dm is elemental mass.

Principal Moment of Inertia

If one can diagonalized Inertia tensor into diagonal matrix then the diagonal element is known as principal moment of inertia

And x, y, z component of angular momentum reduces to

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$J_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$J_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The secular equation of characteristic equationby given by $\left[I - \lambda U\right] = 0$

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

The solution of characteristic equation $\lambda = I_1, I_2, I_3$ are three principal moment of inertia.

Rotational kinetic energy of a rigid body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

Kinetic energy of particle m_i (which is explained in previous topic).

$$T_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

Kinetic energy for entire body

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} \sum \left(\vec{\omega} \times \vec{r}_i \right) \cdot m_i \vec{v}_i$$

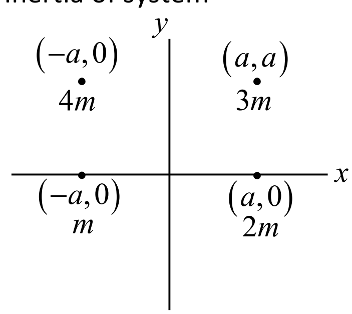
$$= \frac{1}{2} \sum_i \vec{\omega} \cdot \left(\vec{r}_i \times m_i \vec{v}_i \right) = \frac{1}{2} \vec{\omega} \cdot \sum_i \left(\vec{r}_i \times m \vec{v}_i \right)$$

Since, $J = \sum_i \left(\vec{r}_i \times m_i \vec{v}_i \right)$, so $T = \frac{1}{2} \vec{\omega} \cdot \vec{J}$

Kinetic energy in a co-ordinate system of principle axis is given by

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

Example: Find the moment of Inertia of system



Solution: $m_1 = m, \quad m_2 = 2m, \quad m_3 = 3m, \quad m_4 = 4m$

$$(x_1, y_1, z_1) = (-a, 0, 0), (x_2, y_2, z_2) = (a, 0, 0), (x_3, y_3, z_3) = (a, a, 0), (x_4, y_4, z_4) = (-a, a, 0)$$

$$I_{XX} = \sum_i m_i \left(y_i^2 + z_i^2 \right) = m \times 0^2 + 2m \times 0^2 + 3m \times a^2 + 4m \times (-a)^2 = 7ma^2$$

$$I_{YY} = \sum_i m_i \left(x_i^2 + z_i^2 \right) = m \times (-a)^2 + 2m \times a^2 + 3m \times a^2 + 4m \times (-a)^2 = 10ma^2$$

$$I_{ZZ} = \sum_i m_i \left(x_i^2 + y_i^2 \right) = m \times \left((-a)^2 + 0^2 \right) + 2m \times \left(a^2 + 0^2 \right) + 3m \times \left(a^2 + a^2 \right) + 4m \times \left((-a)^2 + a^2 \right) = 17ma^2$$

$$I_{XY} = I_{YX} = -\sum_i m_i x_i y_i = 0 + 0 + (-3ma.a) + (-4m.a.(-a)) = ma^2$$

$$I_{XZ} = I_{ZX} = -\sum_i m_i x_i z_i = 0$$

$I_{YZ} = I_{ZY} = -\sum_i m_i y_i z_i = 0$

The moment of Inertia Tensor for given system is $I = \begin{bmatrix} 7ma^2 & ma^2 & 0 \\ ma^2 & 10ma^2 & 0 \\ 0 & 0 & 17ma^2 \end{bmatrix}$

Example: A rigid body consists of three point masses of $2kg, 1kg$ and $4kg$, connected by massless rods. These masses are located at coordinates $(1,-1,1),(2,0,2)$ and $(-1,1,0)$ in meters, respectively. Compute the inertia tensor of this system. What is the angular momentum vector of this body, if it is rotating with an angular velocity $\omega = 3\hat{i} - 2\hat{j} + 4\hat{k}$?

Solution: We have

$$I_{XX} = \sum_i m_i \left(y_i^2 + z_i^2\right) = 2(1+1) + 1(0+4) + 4(1+0) = 12$$

$$I_{XY} = -\sum_i m_i x_i y_i = -2(-1) - 1(0) - 4(-1) = 6 = I_{YX}$$

$$I_{XZ} = -\sum_i m_i x_i z_i = -2(1) - 1(4) - 4(0) = -6 = I_{ZX}$$

$$I_{YY} = \sum_i m_i \left(x_i^2 + z_i^2\right) = 2(2) + 1(8) + 4(1) = 16$$

$$I_{YZ} = -\sum_i m_i y_i z_i = -2(-1) - 1(0) - 4(0) = 2 = I_{ZY}$$

$$I_{ZZ} = \sum_i m_i \left(x_i^2 + y_i^2\right) = 2(2) + 1(4) + 4(2) = 16$$

Therefore,

$$I = \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix}$$

Given angular velocity can be expressed as

$$\omega = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

We know that, in the matrix form, one can write

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

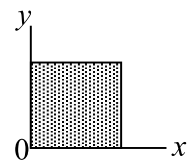
Thus, for this case

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 42 \end{pmatrix}$$

or $L = -6\hat{j} + 42\hat{k}$

Example: (a) Find the moment of inertia of system where square of mass M and area a^2 the origin is at one corner as shown in figure.

(b) Find principle moment of inertia



Solution: $I_{XX} = \int dm\left(y^2 + z^2\right) = \frac{M}{a^2}\int_0^a y^2 dy \int_0^a dx = \frac{M}{a^2} \times \frac{a^3}{3} \times a = \frac{Ma^2}{3}$

$$I_{YY} = \int dm\left(x^2 + z^2\right) = \frac{M}{a^2}\int_0^a x^2 dx \int_0^a dy = \frac{M}{a^2} \times \frac{a^3}{3} \times a = \frac{Ma^2}{3}$$

$$I_{ZZ} = \int dm\left(x^2 + y^2\right) dxdy = \int x^2 dm + \int y^2 dm = \frac{Ma^2}{3} + \frac{Ma^2}{3} = \frac{2}{3}Ma^2$$

$$I_{XY} = I_{XY} = -\int dmxy = -\frac{M}{a^2}\int_0^a xdx \int_0^a ydy = -\frac{M}{a^2} \times \frac{a^2}{2} \times \frac{a^2}{2} = -\frac{Ma^2}{4}$$

$$I_{XZ} = I_{ZX} = -\int dmxz = 0$$

$$I_{YZ} = I_{ZY} = -\int dmyz = 0$$

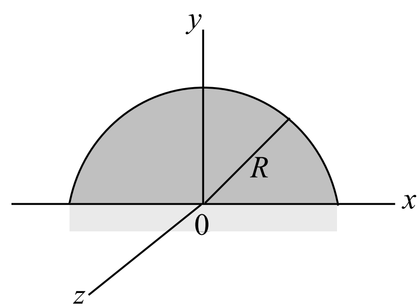
$$I = \begin{bmatrix} \frac{Ma^2}{3} & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3} & 0 \\ 0 & 0 & \frac{2Ma^2}{3} \end{bmatrix}$$

$$[I-\lambda U]=0,\quad I=\begin{bmatrix} \frac{Ma^2}{3}-\lambda & -\frac{Ma^2}{4} & 0 \\ -\frac{Ma^2}{4} & \frac{Ma^2}{3}-\lambda & 0 \\ 0 & 0 & \frac{2Ma^2}{3}-\lambda \end{bmatrix}=0$$

$$\left(\frac{2Ma^2}{3}-\lambda\right)\left(\left(\frac{Ma^2}{3}-\lambda\right)^2-\left(\frac{Ma^2}{4}\right)^2\right)=0$$

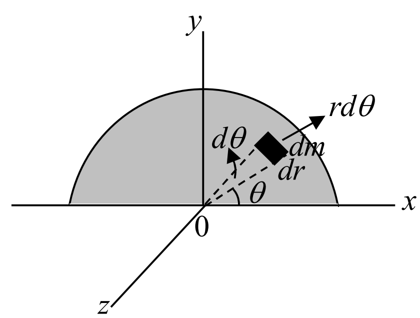
$$\lambda_1=\frac{7Ma^2}{12}\quad,\ \lambda_2=\frac{Ma^2}{12}\quad \lambda_3=\frac{2Ma^2}{3}$$

Example: Find the moment of inertia tensor of semi circular disc of mass M and radius R where origin is at the center of disc.



Solution: We can solve the problem in circular polar coordinate $x=r\cos\theta, y=r\sin\theta, z=0$

$$dm=\frac{M}{\frac{\pi R^2}{2}}rdrd\theta$$



$$I_{XX}=\int\left(y^2+z^2\right)dm\quad \frac{2M}{\pi R^2}\int_0^R\int_0^\pi r^2\sin^2\theta rdrd\theta=\frac{2M}{\pi R^2}\int_0^R r^3dr\int_0^\pi\sin^2\theta d\theta=\frac{2M}{\pi R^2}\times\frac{R^4}{4}\times\frac{\pi}{2}=\frac{MR^2}{4}$$

$$I_{YY}=\int\left(x^2+z^2\right)dm\quad \frac{2M}{\pi R^2}\int_0^R\int_0^\pi r^2\cos^2\theta rdrd\theta=\frac{2M}{\pi R^2}\int_0^R r^3dr\int_0^\pi\cos^2\theta d\theta=\frac{2M}{\pi R^2}\times\frac{R^4}{4}\times\frac{\pi}{2}=\frac{MR^2}{4}$$

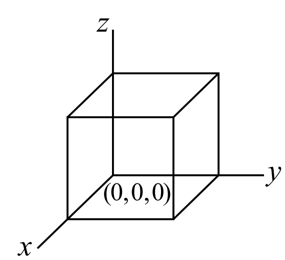
$$I_{ZZ}=\int\left(x^2+y^2\right)dm=\int x^2dm+\int y^2dm=I_{YY}+I_{XX}=\frac{MR^2}{4}+\frac{MR^2}{4}=\frac{MR^2}{2}$$

$$I_{XY}=I_{YX}=-\int xydm=-\frac{2M}{\pi R^2}\int_0^R\int_0^\pi r\cos\theta.r\sin\theta rdrd\theta=-\frac{2M}{\pi R^2}\int_0^R r^3dr\int_0^\pi\sin\theta\cos\theta=0$$

$$I_{XZ}=I_{ZX}=-\int xzdm=0\quad I_{YZ}=I_{ZY}=-\int yzdm=0$$

The moment of inertia tensor $I=\begin{bmatrix}\frac{MR^2}{4}&0&0\\0&\frac{MR^2}{4}&0\\0&0&\frac{MR^2}{2}\end{bmatrix}$

Example: Consider a cube of volume a^3 and mass M, which is situated such the origin O is at one of the corner, consider cube has uniform density ρ. Then,



(a) Find the moment of inertia tensor.

(b) Find the principal moment of inertia tensor.

(c) Find the angular momentum and kinetic energy when the axis of rotation is parallel to $\hat{x}$ [that is, $\omega=(\omega,0,0)$]

(d) Find the angular momentum when the axis of rotation is parallel to is along the body diagonal in the direction $(1,1,1)$.

Solution: (a) $I_{xx}=\int\rho\left(y^2+z^2\right)dV$ where $\rho=\frac{M}{a^3}=\frac{M}{a^3}\int_0^a\int_0^a\int_0^a\left(y^2+z^2\right)dx\,dy\,dz=\frac{2\rho a^5}{3}$

$$\Rightarrow I_{xx}=\frac{2Ma^5}{a^3 3}=\frac{2M}{3}a^2,$$

Similarly,

$$I_{yy}=\int_0^a\int_0^a\int_0^a\rho\left(x^2+z^2\right)dV=\frac{2}{3}Ma^2\quad\text{and}\quad \mathrm{I}_{zz}=\int_0^a\int_0^a\int_0^a\rho\left(x^2+y^2\right)dV=\frac{2}{3}Ma^2$$

$$I_{xy}=-\int_0^a\int_0^a\int_0^a\rho xy\,dx\,dy\,dz=I_{yx}=-\frac{Ma^2}{4}\quad\text{and } I_{xz}=I_{zx}=-\int_0^a\int_0^a\int_0^a\rho xz\,dV=-\frac{1}{4}Ma^2$$

$$I_{yz}=I_{yz}=-\int_0^a\int_0^a\int_0^a\rho yz\,dV=-\frac{1}{4}Ma^2$$

Hence, $I=\frac{Ma^2}{12}\begin{pmatrix}8&-3&-3\\-3&8&-3\\-3&-3&8\end{pmatrix}$.

(b) Principle M.I. is given by characteristic equation

$$\begin{pmatrix} \frac{8Ma^2}{12}-I & -\frac{3Ma^2}{12} & -\frac{3Ma^2}{12} \\ -\frac{3Ma^2}{12} & \frac{8Ma^2}{12}-I & -\frac{3Ma^2}{12} \\ -\frac{3Ma^2}{12} & -\frac{3Ma^2}{12} & \frac{8Ma^2}{12}-I \end{pmatrix} = 0$$

$\Rightarrow I_1 = I_2 = \frac{11}{12}Ma^2$ and $I_3 = \frac{1}{6}Ma^2$

(c) $I = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$

The angular momentum is given by

$$L = I\omega = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \frac{Ma^2}{12}\begin{bmatrix} 8\omega \\ -3\omega \\ -3\omega \end{bmatrix} = \frac{Ma^2}{12}\left(8\omega\hat{i} - 3\omega\hat{j} - 3\omega\hat{k}\right)$$

$$T = \frac{1}{2}\vec{L}.\vec{\omega} = \frac{1}{2}\frac{Ma^2}{12}\left[8\omega \quad -3\omega \quad -3\omega\right]\begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\frac{Ma^2}{12}\times 8\omega^2 = \frac{Ma^2\omega^2}{3}$$

(d) If the cube is rotating about its main diagonal, then the unit vector in the direction of rotation in $u = \left(1/\sqrt{3}\right)(1,1,1)$ and the angular velocity vector is $\omega = \omega u = \left(\omega/\sqrt{3}\right)(1,1,1)$. Thus according to (10.42), the angular momentum for this case is

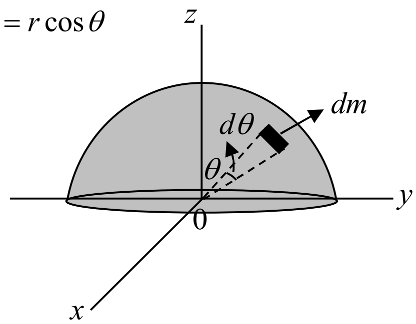
$$L = I\omega = \frac{Ma^2}{12}\frac{\omega}{\sqrt{3}}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{Ma^2}{12}\frac{\omega}{\sqrt{3}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{Ma^2}{6}\vec{\omega}$$

In this case, rotation about the main diagonal of the cube, we see that the angular momentum is in the same direction as the angular velocity.

Example: The solid hemisphere has mass M and radius R. Find the component I_{ZZ} and I_{XY} of inertia tensor

Solution: $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$$dm = \frac{6M}{4\pi R^3} r^2 dr \sin\theta d\theta d\phi$$



$$I_{ZZ} = \int dm\left(x^2 + y^2\right) = \frac{M}{\frac{4\pi R^3}{3}} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} r^2 dr \sin\theta d\theta d\phi \left(r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi\right)$$

$$\frac{6M}{4\pi R^2} \int_0^R r^4 dr \int_0^{\pi/2} \sin^3\theta d\theta \int_0^{2\pi} d\phi = \frac{6M}{4\pi R^3} \times \frac{R^5}{5} \times \frac{2}{3} \times 2\pi = \frac{2}{5} MR^2$$

$$I_{XY} = -\int dm(xy) = -\frac{6M}{4\pi R^3} \int_0^R \int_0^{\pi} \int_0^{2\pi} r^2 dr \sin\theta d\theta d\phi . r\sin\theta\cos\phi r\sin\theta\sin\phi =$$

$$-\frac{6M}{4\pi R^3} \int_0^R r^4 dr \int_0^{\pi/2} \sin^3\theta d\theta \int_0^{2\pi} \cos\phi\sin\phi d\phi = -\frac{6M}{4\pi R^3} \times \frac{2}{3} \times 0 = 0$$