Chapter 1

Stability Analysis and Phase Diagram

4. Phase Portrait

Phase Curve

The curve between position and its conjugate momentum for a given value of energy is known as phase space (curve).

A **phase space** of a dynamical system is a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in the phase space. For mechanical system, the phase space usually consists of all possible values of position and momentum variables.

Separatrix: The part of phase curve that corresponds to energy E which separates (hence the name) the phase space into two distinct areas.

Sign Convention of momentum: When particle is coming from negative x to positive x the momentum has **positive** sign and when particle is coming from positive x to negative x the momentum has **negative** sign.

Turning points: The value of x where Total energy E is equal to Potential energy V. At turning points the kinetic energy will be zero so momentum P_x will become zero.

Properties of phase curve

(1) For given values of energy Phase curve will be unique.

(2) The phase curve will never intersect.

(3) The phase curve can be bounded as well as unbounded.

The phase portrait is a geometrical representation of trajectories of dynamical system in phase plane which is defined by generalized coordinate and its conjugate generalized momentum. The phase trajectory about unstable equilibrium point is mainly unbounded but phase trajectory about stable point is bounded and motion is defined as small oscillation.

How to draw a phase curve:

Step 1: Draw a curve of potential $U(x)\ Vs\ x$, where $U(x)$ as vertical axis and x as horizontal axis.
Step 2: Just below of potential $U(x)$ vs x curve, draw momentum $P(x)$ as vertical axis and x as horizontal axis.

Step 3: For different values of constant energy in $U(x)\ Vs\ x$ draw the trend of $P(x)$ vs x in all classical allowed region.

Step 4: Use sign convention as mention above.

Example: If potential in one dimension is given by $V(x)=-\frac{x^2}{2}+\frac{x^4}{4}$ then plot the phase curve i.e. curve between momentum p_x as function of x for all possible range of energy E.

Solution: To plot phase curve first one should plot potential (V vs x), then on the same axis one should plot momentum with common x axis.

We can check how momentum is changing with position keeping in mind how potential is changing with position.

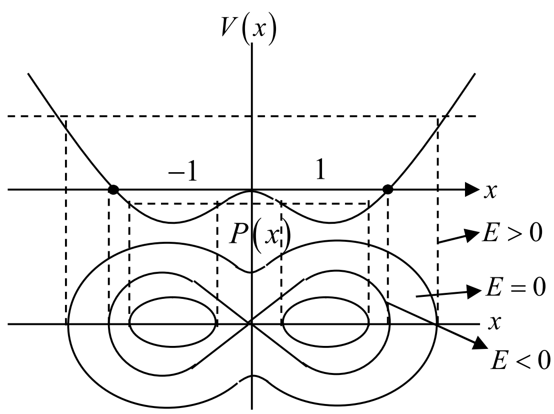
One will plot the phase curve by assuming that if the potential is increasing, then kinetic energy will be decreasing and if the potential is decreasing, then kinetic energy will be increasing because total energy will always remain constant. One should plot the phase curve for different range of energy.

For example in the given potential, there are three range of energy.

Case 1: If $-\frac{1}{4} < E < 0$ the particle has motion about stable equilibrium point $x = 1, -1$ the motion is bounded.

Case 2: If $0 < E < \infty$ the particle has motion about unstable equilibrium point $x = 0$ the motion is bounded.

Case 3: At $E = 0$ the particle can be landed exactly at unstable equilibrium point which is nature of transition from case 1 to case 2.

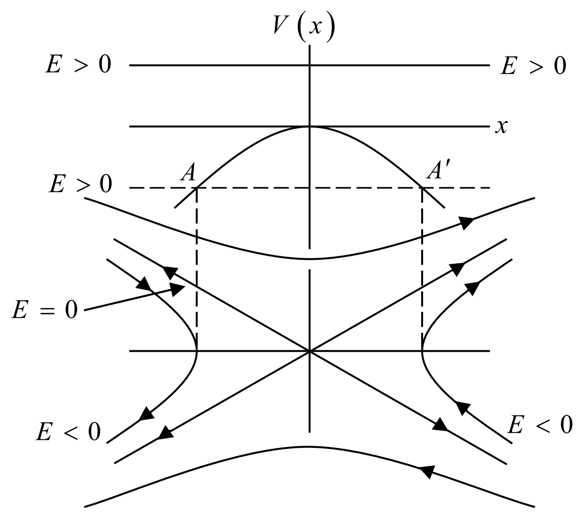


Example: If potential in one dimension is given by $V(x) = -kx^2$ then plot the phase curve i.e. curve between momentum p_x as function of x for all possible range of energy E.

Solution: To plot phase curve first one should plot potential (V vs x), then on the same axis one should plot momentum with common x axis . We can check how momentum is changing with position keeping in mind how potential is changing for a given value of energy. For given value of potential the phase curve is hyperbolic as shown in equation

$$E = \frac{p_x^2}{2m} - kx^2 \Rightarrow \frac{p_x^2}{2mE} - \frac{x^2}{E/k} = 1$$

One will plot the phase curve by assuming that if the potential is increasing, then kinetic energy will be decreasing and if potential is decreasing then kinetic energy will be increasing, because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy.



Case 1: $E<0$, the particle will come from $-\infty$. As it is approaches the potential its kinetic energy as well as momentum decreases finally became zero at turning point A and turn back towards $-\infty$ with increasing kinetic energy and momentum.

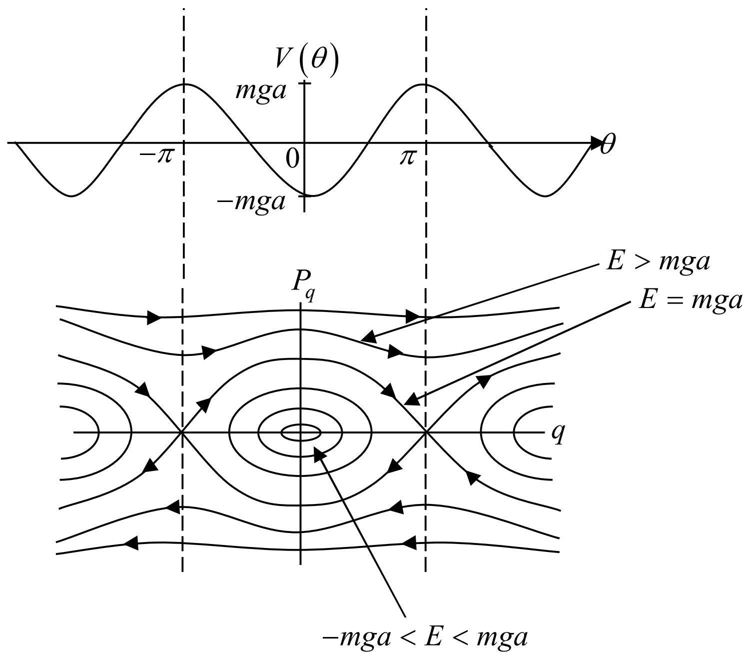
Same trend will also follow when particle approaching the potential from $x=\infty$, for turning point A'.

Case 2: $E>0$, the particle will come from $x=-\infty$. As it approaches the potential. Its kinetic energy as well as momentum decreases till $x=0$. As it crosses $x=0$ and move towards $x=\infty$, again kinetic energy as well as momentum increases and same trend will be followed, when particle approaches the potential to $x=\infty$.

Case 3: $E=0$, the particle can reach at $x=0$, which is unstable equilibrium point and that phase curve will also be separated between $E<0$ and $E>0$, identified as separatix.

$E=\frac{p_x^2}{2m}-kx^2$ for $E=0$ $\Rightarrow p_x \propto \pm x$ which is straight line.

Example: The energy of simple pendulum is given by $E = \frac{p_\theta^2}{2ma^2} - mga\cos\theta$, where p_θ is angular momentum and $-mga\cos\theta$ is potential energy.



One will plot the phase curve by assuming that if the potential energy is increasing, then kinetic energy will be decreasing and if the potential energy is decreasing then kinetic energy will be increasing, because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy.

The stable equilibrium point is $\theta = 0$. $\theta = -\pi$ and $\theta = \pi$ are unstable equilibrium points.

Case 1: For energy $-mga < E < mga$ particle is bounded about stable equilibrium point .so phase curve is periodic.

Case 2: For energy $E > mga$ motion will become unbounded and phase curve will be a periodic. Liberation will take place.

Case 3: For energy $E = mga$ particle will reach at unstable equilibrium point it also separate two type of motion (mention in case 1 and case 2) identified as separatrix.