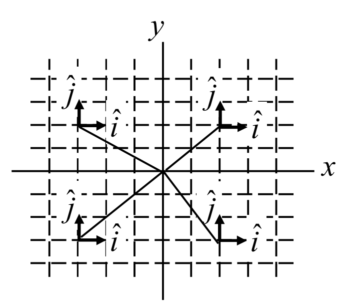
Chapter 2

Newton's Laws of Motion

4. The Motion in Two Dimensional in Polar Coordinate

Two-dimensional motion in Cartesian coordinate

The position vector in two dimension (x, y) plane is given by $r = x\hat{i} + y\hat{j}$ where $\hat{i}, \hat{j}$ are unit vector in x and y direction respectively.

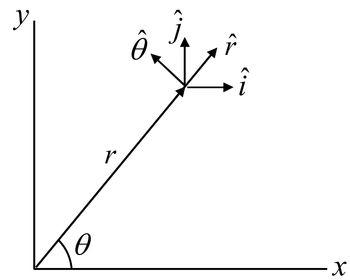


The base unit vector $\hat{i}$ and $\hat{j}$ are not vary with position as shown in figure.

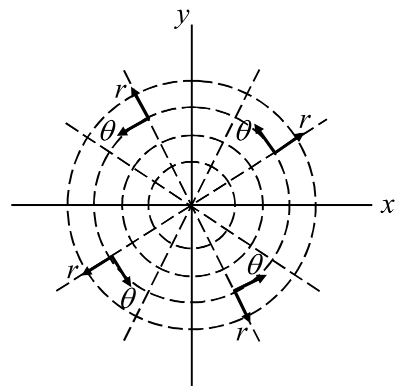
The velocity is given by $v = r = x\hat{i} + y\hat{j}$ and acceleration is given by $a = r = x\hat{i} + y\hat{j}$

Newton's law can be written as $mx = F_x$ and $my = F_y$.

Two-dimensional motion in polar coordinate



Two dimensional system also can be represent in polar coordinate with variable (r,θ) with transformation rule $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$ where r identified as magnitude of vector and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and θ is angle measured from x axis in anti clock wise direction as shown in figure.



In polar coordinate system $\hat{r}$ and $\hat{\theta}$ are unit vector in radial direction and tangential direction of trajectory. One can see from the figure the $\hat{r}$ and $\hat{\theta}$ are vary with position, where $|\hat{\mathbf{r}}| = 1, |\hat{\theta}| = 1$ and $\hat{r}.\hat{\theta} = 0$ conclude they are orthogonal in nature.

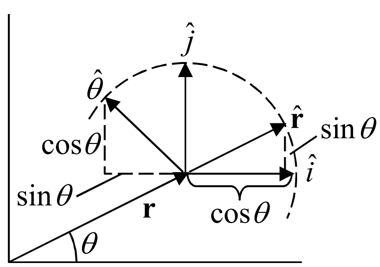
The unit vector $\hat{r}$ and $\hat{\theta}$ can be written in basis of unit vector $\hat{i}$ and $\hat{j}$.

The unit vectors $\hat{i}, \hat{j}$ and $\hat{r}, \hat{\theta}$ at a point in the xy-plane. We see that the orthogonality of $\hat{\mathbf{r}}$ and $\hat{\theta}$ plus the fact that they are unit vectors,

$|\hat{\mathbf{r}}| = 1, |\hat{\theta}| = 1$, $\hat{\mathbf{r}} = \hat{i}\cos\theta + \hat{j}\sin\theta$ and $\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$ which is shown.

The transformation can be shown by rotational Matrix

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$



Time evolution of $\hat{r}$ and $\hat{\theta}$

$$\hat{\mathbf{r}} = \hat{i}\cos\theta + \hat{j}\sin\theta \Rightarrow \frac{d\hat{r}}{dt} = -\hat{i}\sin\theta\theta + \hat{j}\cos\theta\theta \Rightarrow \frac{d\hat{r}}{dt} = \theta\hat{\theta}$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta \Rightarrow \frac{d\hat{\theta}}{dt} = -\hat{i}\cos\theta\theta - \hat{j}\sin\theta\theta \Rightarrow \frac{d\hat{\theta}}{dt} = -\hat{r}\theta$$

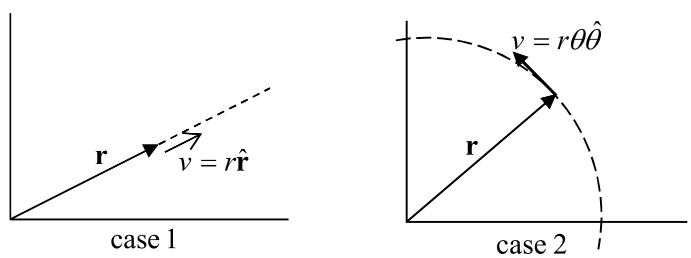
One can easily see unit vector $\hat{r}$ and $\hat{\theta}$ are vary with time.

The Position Vector in Polar Coordinate

$$r = x\hat{i} + y\hat{j}\,,\; r = |r|\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right] \Rightarrow r = |r|\,\hat{r}$$

$\mathbf{r} = r\,\hat{\mathbf{r}}$ is sometimes confusing, because the equation as written seems to make no reference to the angle θ. We know that two parameters needed to specify a position in two dimensional space (in Cartesian coordinates they are x and y), but the equation $\mathbf{r} = r\,\hat{\mathbf{r}}$ seems to contain only the quantity r. The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is origin. Although θ does not occur explicitly in $r\,\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r} = r\,\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ. However, by common conversation $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

Velocity Vector in Polar Coordinate



$$v = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = r\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \;\Rightarrow v = r\hat{r} + r\theta\hat{\theta}$$

where r is radial velocity in $\hat{r}$ direction and $r\theta$ is tangential velocity in $\hat{\theta}$ direction as shown in figure and the magnitude to velocity vector $|v| = \sqrt{r^2 + r^2\theta^2}$

Acceleration Vector in Polar Coordinate

$$\frac{dv}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\theta\hat{\theta} + r\frac{d\theta}{dt}\hat{\theta} + r\theta\frac{d\hat{\theta}}{dt}$$

$$\frac{dv}{dt} = r\hat{r} + r\theta\hat{\theta} + r\theta\hat{\theta} + r\theta\hat{\theta} + r\theta(-\theta)\hat{r}$$

$$a = (r - r\theta^2)\hat{r} + \left(r\theta + 2r\theta\right)\hat{\theta} \;\Rightarrow a = a_r\hat{r} + a_\theta\hat{\theta}$$

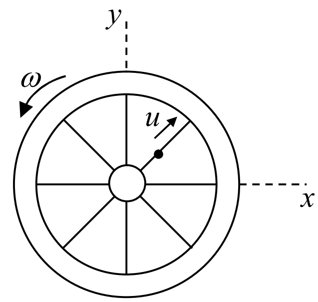
$a_r = r - r\theta^2$ is radial acceleration and $a_\theta = r\theta + 2r\theta$ is tangential acceleration .

So Newton’s law in polar coordinate can be written as

$F_r = ma_r = m(r - r\theta^2)$ where F_r is external force in radial direction .

$F_\theta = ma_\theta = m(r\theta + 2r\theta)$ where F_θ is external force in tangential direction.

Example: A bead moves along the spoke of wheel at constant speed u meter per second .The wheel rotates with uniform angular velocity $\theta = \omega$ radians per second about an axis fixed in space. At $t = 0$ the spoke is along the x axis and bead is at the origin.



(a) Find the velocity of particle

(b) Find the acceleration of particle

Solution: $r = ut \Rightarrow r = u$, $\theta = \omega$

(a) $v = r\hat{r} + r\theta\hat{\theta} \Rightarrow v = u\hat{r} + ut\omega$

(b) $a = \left(r - r\theta^2\right)\hat{r} + \left(r\theta + 2r\theta\right)$

$r = ut \Rightarrow r = u, r = 0$, $\theta = \omega \Rightarrow \theta = 0$

$a = -ut\omega^2\hat{r} + 2u\omega\hat{\theta}$

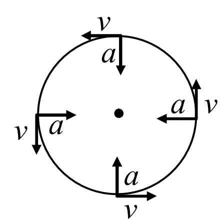
Circular Motion

For circular motion for radius r_0, at $r = r_0$, then $r = 0$ so $F_r = ma_r = -mr_0\theta^2$, where F_r is force in radial direction and $F_\theta = ma_\theta = mr_0\theta$, where F_θ is force in tangential direction.

There are two type of circular motion.

(1) **Uniform Circular Motion**

If there is not any force in tangential direction $F_\theta = 0$ is condition, then motion is uniform circular motion i.e., $\theta = \omega$ is constant known as angular speed and tangential speed is given by $v = r_0\omega$.

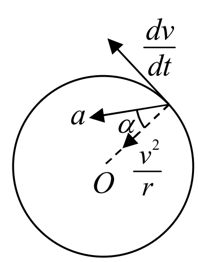


(2) **Non-uniform Circular Motion**

For non-uniform circular motion of radius r radial acceleration is

$$a_r = -mr\theta(t)^2 = -mr\omega(t)^2 = -\frac{mv(t)^2}{r}$$ and

tangential acceleration $a_\theta = \frac{dv}{dt}$



the acceleration for non uniform circular motion is given by $a = a_r\hat{r} + a_\theta\hat{\theta} \Rightarrow -\frac{mv^2}{r}\hat{r} + \frac{dv}{dt}\hat{\theta}$

the magnitude of acceleration is given by $|a| = \sqrt{a_r^2 + a_\theta^2} \Rightarrow \sqrt{\left(\frac{mv^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$

Example: Find the magnitude of the linear acceleration of a particle moving in a circle of radius $10\,cm$ with uniform speed completing the circle in $4\,s$.

Solution: The distance covered in completing the circle is $2\pi r = 2\pi \times 10\,cm$. The linear speed is

$$v = \frac{2\pi r}{t} = \frac{2\pi \times 10\,cm}{4\,s} = 5\,\pi\,cm/s$$

The linear acceleration is

$$a = \frac{v^2}{r} = \frac{(5\pi\,cm/s)^2}{10\,cm} = 2.5\,\pi^2 cm/s^2$$

This acceleration is directed towards the centre of the circle.

Example: A particle moves in a circle of radius $20\,cm$. Its linear speed is given by $v = 2t$, where t is in second and v in metre/second. Find the radial and tangential acceleration at $t = 3\,s$.

Solution: The linear speed at $t = 3\,s$ is $v = 2t = 6\,m/s$

The radial acceleration at $t = 3\,s$ is

$$a_r = v^2/r = \frac{36\,m^2 s^2}{0.20\,m} = 180\,m/s^2$$

The tangential acceleration is $a_t = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2\,m/s^2$

Calculation of torque and angular momentum

$$F = F_r\hat{r} + F_\theta\hat{\theta}$$

Torque is given by $\tau = r \times F = r\hat{r} \times F = m\left(r^2\theta + 2rr\theta\right)\hat{r} \times \hat{\theta} = m\left(r^2\theta + 2rr\theta\right)\hat{z}$

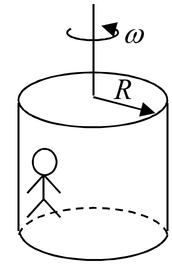
Torque is also defined rate of change of momentum $\tau = \frac{dJ}{dt} = m\left(r^2\theta + 2rr\theta\right) = \frac{d\left(mr^2\theta\right)}{dt}$

So, angular momentum is given by $J = mr^2\theta$

If $F_\theta = 0 \Rightarrow m\left(r\theta + 2r\theta\right) = 0 \Rightarrow \frac{d\left(mr^2\theta\right)}{dt} = 0 \Rightarrow mr^2\theta =$ constant

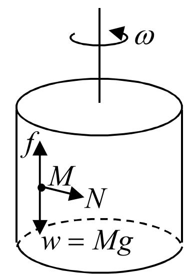
If there is not any tangential force is in plane then angular momentum of the system is conserve.

Example: The Spinning Terror



The Spinning Terror is an amusement park ride – a large vertical drum which spins so fast that everyone inside stays pinned against the wall when the floor drops away. What is the minimum steady angular velocity ω, which allows the floor to be dropped away safely?

Solution: Suppose that the radius of the drum is R and the mass of the body is M. Let μ be the coefficient of friction between the drum and M. The forces on M are the weight W, the friction force f and the normal force exerted by the wall, N as shown below. The radial acceleration is $R\omega^2$ toward the axis, and the radial equation of motion is $N = MR\omega^2$



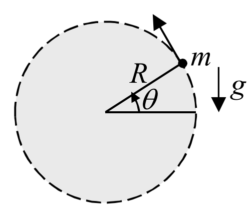
By the law of static friction, $f \le \mu N = \mu MR\omega^2$

Since, we require M to be in vertical equilibrium, $f = Mg$,

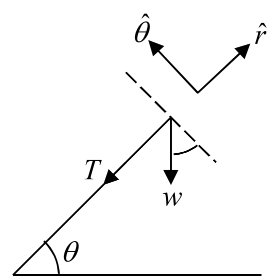
and we have, $Mg \le \mu MR\omega^2$ or $\omega^2 \ge \dfrac{g}{\mu R}$

The smallest value of ω that will work is $\omega_{\min} = \sqrt{\dfrac{g}{\mu R}}$

Example: Mass m is whirled on the end of string R the motion is in a vertical plane in the gravitational field of earth .write down the equation of motion in polar coordinate.



Solution: There are only two forces acted on body one tension T in radial direction and weight mg in vertical direction. Taking component of these two forces in $\hat{r}$ and $\hat{\theta}$ direction.

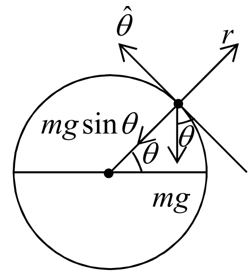


The Newton's law of motion in radial direction is

$$F_r = ma_r \text{ in } \hat{r} \text{ direction} \Rightarrow -T - mg\sin\theta = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

The Newton's law of motion in tangential direction is

$$F_\theta = ma_\theta \Rightarrow -mg\cos\theta = m(r\theta + 2r\theta)$$

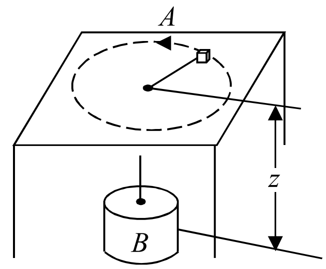


The equation of constrain is $r = R, \quad r = 0, \quad r = 0$

$$-T - mg\sin\theta = -mR\theta^2 \text{ in } \hat{r} \text{ direction}$$

$$-mg\cos\theta = mR\theta \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{R}\cos\theta \text{ in } \hat{\theta} \text{ direction}$$

Example: A horizontal frictionless table has a small hole in the centre of table .Block A of mass m_a on the table is connected by a block B of mass m_b hanging beneath by a string of negligible mass can move under gravity only in vertical direction, which passes through the hole as shown in figure.



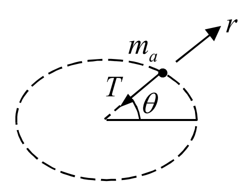
(a) Write down equation of motion in radial, tangential and vertical direction.

(b) What is equation of constrain?

(c) What will be the acceleration of B when A is moving with angular velocity ω at radius r_0 ?

Solution: Suppose of block A rotating in circle with angular velocity ω of radius r_0 what is acceleration of block B assume $\hat{z}$ direction is shown in figure.

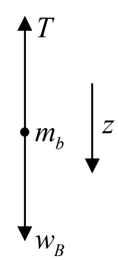
The external force in radial direction is tension, not any external force in tangential direction and weight of body m_b and tension in vertical direction.



(a) $\quad m_a(r - r\theta^2) = -T$ in $\hat{r}$ direction ....(1)

$m_a(r\theta + 2r\theta) = 0$ in $\hat{\theta}$ direction ....(2)

$m_a z = m_b g - T$ in $\hat{z}$ ....(3)



(b) Since length of the rope is constant then

$r + z = l \Rightarrow r = -z$ ....(4)

(c) Put value of $r = -z$, $T = m_a(z + r\theta^2)$

$$z = \frac{m_b g - m_a r\theta^2}{m_b + m_a} \text{ put } r = r_0, \theta = \omega \Rightarrow z = \frac{m_b g - m_a r_0\omega^2}{m_a + m_b}$$

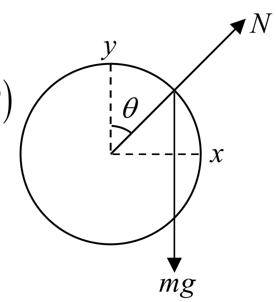
Example: A bead rests at the top of a fixed frictionless hoop of radius R lies in vertical plane.

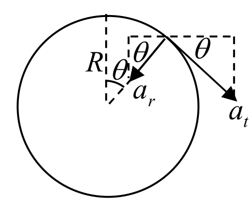
(a) At particular instance particle make angle θ with vertical Write down equation of motion in polar coordinate.

(b) Find the value of $\theta = \theta_c$ such that particle will just leave the surface of ring.

(c) At what angle $\theta = \theta_p$ the acceleration of the particle in vertical direction. What is relation between θ_c and θ_p.

Solution:





(a) $N - mg\cos\theta = ma_r = m\left(r - r\theta^2\right)$ and $mg\sin\theta = ma_\theta = m\left(r\theta + 2r\theta\right)$

Equation of constrain $r = R, r = 0, r = 0$

$N - mg\cos\theta = -mr\theta^2 = ma_r$, $mg\sin\theta = mR\theta = ma_\theta$

(b) $\theta = \theta_c$ $N = 0$ $-mg\cos\theta_c = -mr\theta^2 = -\frac{mv^2}{R}$

From conservation of energy

$$mg2R = \frac{1}{2}mv^2 + mg\left(R + R\cos\theta_c\right) \Rightarrow mv^2 = mg2R\left(1 - \cos\theta_c\right)$$

$$mg\cos\theta_c = \frac{mg2R\left(1 - \cos\theta_c\right)}{R} \Rightarrow \cos\theta_c = \frac{2}{3}$$

(c) $a_r = 2g\left(1 - \cos\theta_P\right)$ and $a_\theta = g\sin\theta_p$

If net acceleration is in vertical direction then horizontal component must be cancelled to each other.

$$a_r\sin\theta_p = a_\theta\cos\theta_p \Rightarrow 2g\left(1 - \cos\theta_p\right)\sin\theta_p = g\sin\theta_p\cos\theta_p \Rightarrow \cos\theta_p = \frac{2}{3}$$

$$\theta_c = \theta_p = \cos^{-1}\frac{2}{3}$$