Chapter 6

Linear Momentum And Energy

10. Two-Dimensional Motion

Oblique Collision:

Let's now look at the more general case of two-dimensional motion. Three-Dimensional motion is just more of the same, so we'll confine ourselves to 2-D. Everything is basically the same as in 1-D, except that there is one more momentum equation, and one more variable to solve for. This is best seen through an example.

Elastic Collision in Two Dimensions

Consider two objects A and B of mass m_1 and m_2 kept on the X - axis (figure). Initially, the object B is at rest and A moves towards B with a speed u_1. If the collision is not head -on (the force during the collision is not along the initial velocity), the objects move along different lines. Suppose the object A moves with a velocity $\vec{v}$ making an angle θ with the X - axis and the object B moves with a velocity $\vec{v}_2$ making an angle ϕ with the same axis. Also, suppose $\vec{v}_1$ and $\vec{v}_2$ lie in X - Y plane. Using conservation of momentum in X and Y directions, we get

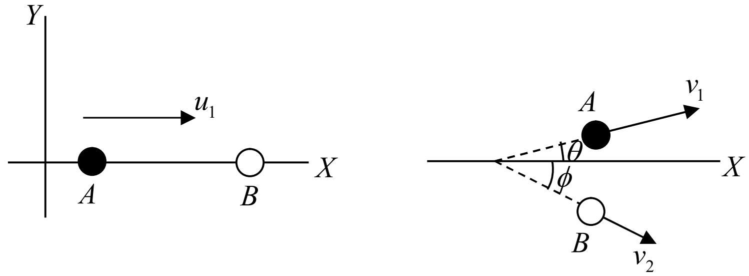
$$m_1u_1 = m_1v_1\cos\theta + m_2v_2\cos\phi \qquad \text{(i)}$$

and $$0 = m_1v_1\sin\theta - m_2v_2\sin\phi \qquad \text{(ii)}$$

If the collision is elastic, the final kinetic energy is equal to the initial kinetic energy. Thus,

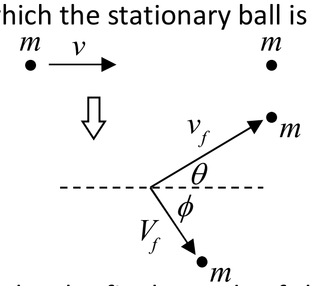
$$\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad \text{(iii)}$$

We have four unknowns v_1, v_2, θ and ϕ to describe the final motion whereas there are only three relations. Thus, the final motion cannot be uniquely determined with this information.



Example: A billiard ball with speed v approaches an identical stationary one. The balls bounce off each other elastically, in such a way that the incoming one gets deflected by an angle θ (see figure). What are the final speeds of the balls?

What is the angle ϕ at which the stationary ball is deflected?



Solution: Let v_f and V_f be the final speeds of the balls. Then conservation of p_x, p_y, and E give, respectively.

$$mv = mv_f\cos\theta + mV_f\cos\phi$$

$$0 = mv_f\sin\theta - mV_f\sin\phi$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}mV_f^2$$

We must solve these three equations for the three unknowns v_f, V_f , and ϕ. There are various ways to do this. Here's one. Eliminate ϕ by adding the squares of the first two equations (after putting the v_f terms on the left-hand side) to obtain $v^2 - 2vv_f\cos\theta + v_f^2 = V_f^2$

Now eliminate V_f by combining this the third equation to obtain $v_f = v\cos\theta$

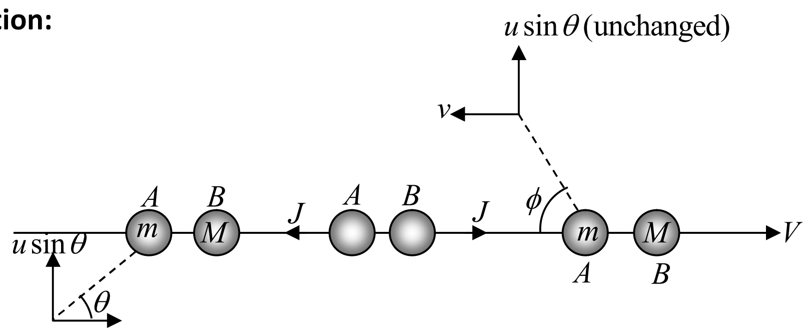
The third equation then yields $V_f = v\sin\theta$

The second equation then gives $m(v\cos\theta)\sin\theta = m(v\sin\theta)\sin\phi$; which implies $\cos\theta = \sin\phi$ (or $\theta = 0$, which corresponds to no collision). Therefore, $\phi = 90^0 - \theta$

In other words, the balls bounce off at right angles with respect to each other. This fact is well known to pool players. Problem 5.19 gives another (cleaner) way to demonstrate this result. Note that we needed to specify one of the four quantities, v_f, V_f, θ, ϕ. (we chose θ), because we have only three equations. Intuitively, we can't expect to solve for all four of these quantities, because we can imagine one ball hitting the other at various distances away from directly head-on, which will cause the balls to be deflected at various angles.

Example: Two smooth spheres A and B, or equal radius but masses m and M, are free to move on a horizontal table A is projected with speed u towards B which is at rest. On impact, the line joining their centres is inclined at an angle θ to the velocity of A before impact. if e is the coefficient of restitution between the spheres, find the speed with which B begins to move. If A's path after impact is perpendicular to its path before impact, show that $\tan^2\theta = \frac{eM - m}{M + m}$

Solution:



When B is struck by the impulse J, it begins to move in the direction of J as shown in the diagram. Along the line of centre, we apply

(a) Conservation of linear momentum,

i.e., $mu\cos\theta = MV - mv$ (i)

(b) Law of restitution,

i.e., $eu\cos\theta = V + v$ (ii)

Solving equations, (i) and (ii), we get

$v = \dfrac{(eM - m)u\cos\theta}{(M+m)}$ and $V = \dfrac{(1+e)mu\cos\theta}{M+m}$

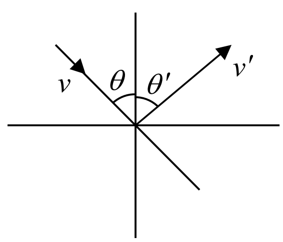
Hence, $\tan\phi = \dfrac{u\sin\theta}{v} = \dfrac{(M+m)\tan\theta}{(eM-m)}$

Example: A ball of mass m hits a floor with a speed v making an angle of incidence θ with the normal. The coefficient of restitution is e.

(a) Find the speed of the reflected ball .

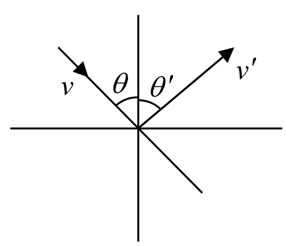
(b) Find the angle of reflection of the ball.

(c) Find the angle of reflection for elastic collision



Solution: (a) Suppose the angle of reflection is θ' and the speed after the collision is v'. The floor exerts a force on the ball along the normal during the collision. There is no force parallel to the surface. Thus, the parallel component of the velocity of the ball remains unchanged. This gives

$$v'\sin\theta' = v\sin\theta \quad \text{(i)}$$



For the components normal to the floor,

The velocity of separation $= v'\cos\theta'$

and the velocity of approach $= v\cos\theta$

Hence, $v'\cos\theta' = ev\cos\theta$ (ii)

From (i) and (ii), $v' = v\sqrt{\sin^2\theta + e^2\cos^2\theta}$

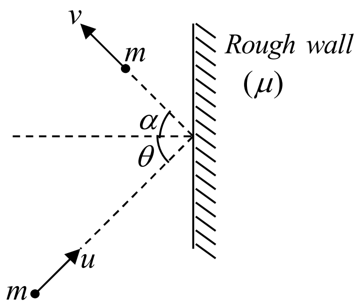
(b) Dividing equation (i) and (ii), we get $\tan\theta' = \dfrac{\tan\theta}{e}$

(c) For elastic collision, $e = 1$ so that $\theta' = \theta$ and $v' = v$

Note: If collision is perfectly elastic, no energy is absorbed by the floor and hence $e = 1$

So $\tan\theta = \tan\theta' \Rightarrow \theta = \theta'$ and $v' = v$

Example: A small ball of mass m collides with a rough wall having coefficient of friction μ at an angle θ with the normal to the wall. If after collision the ball moves with angle α with the normal to the wall and the coefficient of restitution is e then find the reflected velocity v of the ball just after collision.



Solution: $mv\cos\alpha - \left(m\left(-u\cos\theta\right)\right) = \int Ndt$

$$mv\sin\alpha - mu\sin\theta = -\mu\int Ndt$$

and $e = \dfrac{v\cos\alpha}{u\cos\theta} \Rightarrow v\cos\alpha = eu\cos\theta$

or $mv\sin\alpha - mu\sin\theta = -\mu\left(mv\cos\alpha + mu\cos\theta\right)$

or $v = \dfrac{u}{\sin\alpha}\left[\sin\theta - \mu\cos\theta\left(e+1\right)\right]$