Chapter 6

Linear Momentum And Energy

2. Conservation of Momentum

In the last section we found that the total external force F acting on a system is related to the total momentum P of the system by $\vec{F} = \frac{d\vec{p}}{dt}$

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case $\vec{F} = 0$, and $\frac{d\vec{p}}{dt} = 0$. The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system are constant. This is the law of conservation of momentum. If external force is zero then momentum of system is conserve and center of mass of system is not changing.

The Center of Mass Frame

Consider a inertial frame S and another inertial frame $S^{'}$ that at constant velocity $\vec{v}$ with respect to S .system of particle is moving with respect to $S^{'}$ is $\vec{u}_i^{'}$ .Same system of particles have velocity with respect to S is $\vec{u}_i^{'}$

From Galilean transformation $\vec{u}_i = \vec{u}_i^{'} + \vec{v}$

Let us consider the unique frame in which the total momentum of a particles is zero. This is called the center of mass frame or $C.M$ frame

If Total momentum from is $\vec{p} = \sum_i m_i u_i$ from S frame then the center of mass is $S^{'}$ that moves with velocity $\vec{v} = \frac{\vec{p}}{M} = \frac{\sum_i m_i u_i}{M}$ where $M = \sum_i m_i$ is total mass

So from $S^{'}$ or center of mass frame the momentum is

$$\vec{p}^{'} = \sum_i m_i u_i^{'} = \sum_i m_i \left(\vec{u}_i - \vec{v}\right) = \sum_i m_i \left(\vec{u}_i - \frac{\vec{p}}{M}\right) = \vec{p} - \vec{p} = 0$$

So center of mass frame is known as zero momentum frame.

Example: A mass m with speed v approaches a stationary particle of mass M

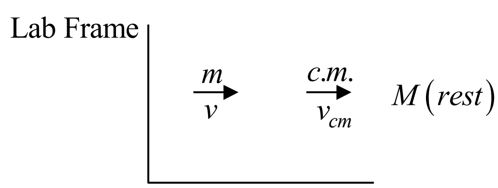
(a) Find the velocity of center of mass with respect to laboratory

(b) Find the velocity of particle with respect to center of mass before collision.

(c) Find the velocity of particles after collision such that center of mass has zero momentum i.e. with respect to center of mass.

(d) Find the velocity of particles just after collision with respect to laboratory.

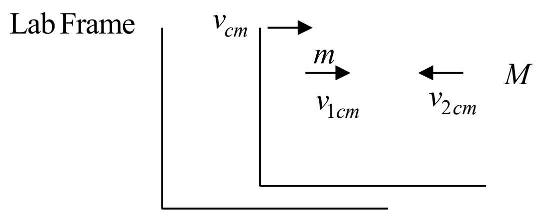
Solution: (a) $m_1 = m \quad m_2 = M \quad v_1 = v$ and $v_1 = 0$



$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \Rightarrow \frac{mv}{m+M}$$

(b) $v_{m,cm} = v_1 - v_{cm} = v - \frac{mv}{m+M} = \frac{Mv}{m+M}$

$$v_{M,cm} = v_2 - v_{cm} = 0 - \frac{mv}{m+M} = -\frac{mv}{m+M}$$



(c) If momentum of center of mass is zero then velocity of mass m with respect to center of mass is

$$v_{m,cm} - \frac{mv}{m+M} = \frac{Mv}{m+M} - \frac{mv}{m+M} = \left(\frac{M-m}{M+m}\right)v$$

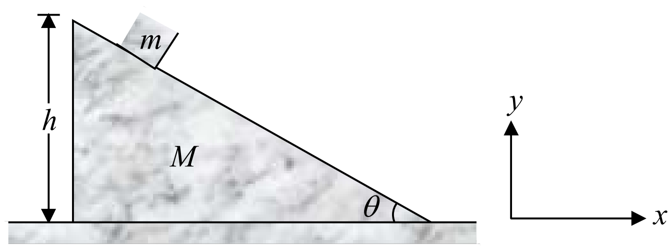
If momentum of center of mass is zero then velocity of mass M with respect to center of mass is

$$v_{m,cm} - \frac{mv}{m+M} = -\frac{mv}{m+M} - \frac{mv}{m+M} = -\frac{2m}{m+M}v$$

(d) Then velocity of particle m after collision with respect $-\left(\frac{M-m}{M+m}\right)v = \left(\frac{m-M}{M+m}\right)v$

Then velocity of particle m after collision with respect $-\left(-\frac{2m}{M+m}\right)v = \left(\frac{2m}{M+m}\right)v$

Example: A block of mass m slides down frictionless wedge of mass M, when block will reach the bottom how much horizontal distance wedge will move.



Solution: Since there is not any force in horizontal direction, then momentum in horizontal direction is conserved.

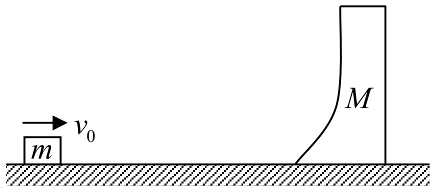
Therefore, center of mass in horizontal distance will not change.

x distance of mass M move towards left with respect to surface. Same time m will move $h\cot\theta - x$ with respect to surface towards right.

Let us assume center of mass is at origin so $\frac{Mx_1 + mx_2}{M+m} = 0$

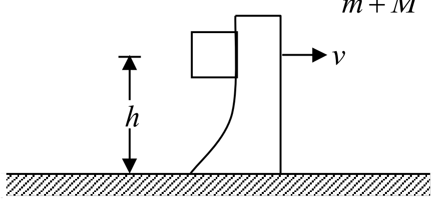
put $x_1 = -x, x_2 = h\cot\theta - x$ so $\frac{-Mx + m\left(h\cot\theta - x\right)}{M+m} = 0 \Rightarrow x = \frac{mh\cot\theta}{m+M}$

Example: All the surface shown in figure are smooth wedge of mass M is free to move. Block of mass m is given a horizontal velocity v_0 as shown in figure. Find the maximum height h attained by m.



Solution: Since there is not any force in horizontal direction so momentum in horizontal direction is conserved.

From conservation of momentum $mv_0 = (m+M)v \Rightarrow v = \dfrac{mv_0}{m+M}$

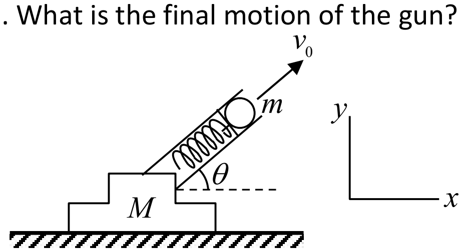


From conservation of energy $\frac{1}{2}mv_0^2 = \frac{1}{2}(m+M)v^2 + mgh$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m+M)\frac{(mv_0)^2}{(m+M)^2} + mgh \Rightarrow \frac{1}{2}mv_0^2 = \frac{1}{2}\frac{m^2v_0^2}{(m+M)} + mgh$$

$$\frac{1}{2}mv_0^2 - \frac{1}{2}\frac{m^2v_0^2}{(m+M)} = mgh \Rightarrow mgh = \frac{1}{2}mv_0^2\left(1 - \frac{m}{m+M}\right) \Rightarrow h = \frac{v^2}{2g}\frac{M}{m+M}$$

Example: A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m and the muzzle velocity of the marble is v_0. What is the final motion of the gun?



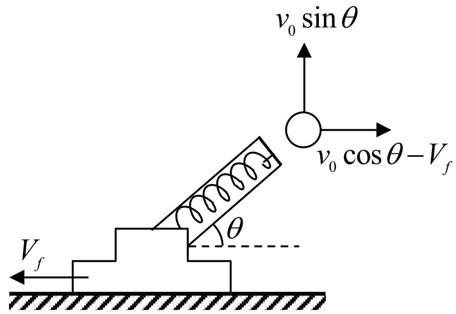
Solution: Take the physical system to be the gun and marble, Gravity and the normal force of the table act on the system. Both these forces are vertical. Since, there are no horizontal external forces.

The x component of the vector equation $F = dP/dt$ is

$$0 = \frac{dP_x}{dt} \qquad (1)$$

According to equation (1) P_x is conserved:

$$P_{x,\text{initial}} = P_{x,\text{final}} \qquad (2)$$

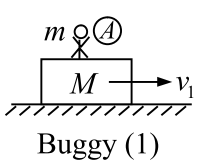


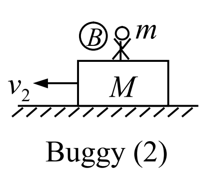
Let the initial time be prior to firing the gun. Then $P_{x,\text{initial}} = 0$. Since the system is initially at rest. After the marble has left the muzzle, the gun recoils with some speed V_f, and its final horizontal momentum is MV_f, to the left. Finding the final velocity of the marble involves a subtle point, however. Physically, the marble's acceleration is due to the force of the gun, and the gun's recoil is due to the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so that at the instant the marble and the gun part company, the gun has its final speed V_f. At that same instant the speed of the marble relative to the gun is v_0. Hence, the final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$. By conservation of horizontal momentum. We have,

$$0 = m(v_0 \cos\theta) - mV_f - MV_f \Rightarrow V_f = \frac{mv_0 \cos\theta}{M + m}$$

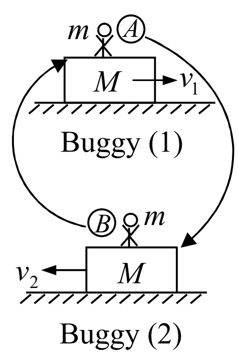
Example: Two identical buggies 1 and 2 with one man in each, move without friction due to inertia along the parallel rails towards each other. When the buggies get opposite to each other, the men exchange their places by jumping in the direction perpendicular to the motion direction. As a consequence, buggy 1 stops and buggy 2 keeps moving in the same direction, with its velocity becoming equal to v. Find the initial velocities of the buggies v_1 and v_2 if the mass of each buggy (without a man) equals M and the mass of each man is m.

Solution: (i) Initial condition of the buggies:

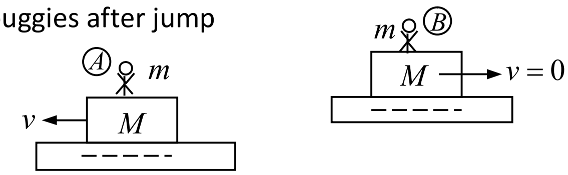




(ii) Status of buggies during jump:



(iii) Status of buggies after jump



During this exchange momentum will be conserved because there is no force is horizontal direction.

Conservation of momentum for buggy (1) $Mv_1 - mv_2 = 0$ ...(i)

Conservation of momentum for buggy (2) $Mv_2 - mv_1 = (m+M)v$ ...(ii)

From (i) and (ii)

$$v_2 = \frac{Mv}{M-m} \qquad \Bigg| \qquad v_1 = \frac{mv}{M-m}$$

But in term of vector: v_2 has opposite direction as v_1.

Then $v_2 = \frac{Mv}{M-m}$ and $v_1 = \frac{-mv}{M-m}$